CONCERNING THE EXISTENCE AND CONSTRUCTION

OF ORTHOGONAL DESIGNS

by

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ABSTRACT

This thesis is concerned with the existence problem for orthogonal designs, amicable orthogonal designs and product designs.

In order to give a basis for attacking the existence problem of orthogonal designs, the problem is solved completely for the case of orthogonal designs of order 16. The ideas developed here are extended to prove that there is no orthogonal design of order n, $n > 40$, and type $(1, 1, 1, 1, 1, n-5)$.

Some properties of amicable orthogonal designs are investigated, especially from the point of view of non-existence. We prove, for example, that there are no amicable orthogonal designs of order $n \equiv 0 \pmod 8$ and types $\big((1); (1, a, n-a-1)\big)$, $a = 2, 3, 4$ or 5.

Product designs are defined and various properties of these designs are given. It is shown how these designs may be combined with amicable orthogonal designs to produce new orthogonal designs. Examples are given of how these designs may be used to produce orthogonal designs of orders 32, 64 and 128. An orthogonal design of order 2^t and type $\left(1, 1, 1, 1, 2, 2, 4, 4, \ldots, 2^{t-2}, 2^{t-2}\right)$ is constructed. This design often meets the Radon bound for the number of variables.

Some properties of Turyn sequences are investigated, and it is shown how these sequences may be used to construct orthogonal designs.

PUBLICATIONS

1. "A non-existence theorem for orthogonal designs", *Utilitas Math.* 10 (1976), 179-184.
2. "Amicable orthogonal designs", *Bull. Austral. Math. Soc.* 14 (1976), 303-314.
3. "The existence of orthogonal designs of order 16 ".
4. "Orthogonal designs in order sixteen", *Combinatorial Mathematics IV: Proc. Fourth Australian Conference.*
5. "Orthogonal designs in order 24", *Combinatorial Mathematics V: Proc. Fifth Australian Conference.*
6. "Using product designs to construct orthogonal designs".
7. "A note on using sequences to construct orthogonal designs", *Colloquia Mathematica Societatis Janos Bolyai* (with Jennifer Seberry Wallis).
8. "An algorithm for orthogonal designs", *Proc. Fifth Manitoba Conf. on Numerical Math.*, 1975, 279-292 (with Peter Eades, Jennifer Seberry Wallis and Ian S. Williams).

CONTENTS

CHAPTER 1

INTRODUCTION

DEFINITION 1.1. An *Hadamard matrix*, H, is an $n \times n$ matrix with entries from $\{\pm 1\}$ satisfying

$$H.H^t = nI_n .$$

Hadamard matrices have found their way into such fields as statistics, engineering and communication theory, and much research has been concerned with their existence. It is conjectured that Hadamard matrices exist for orders 1, 2 and $4t$, for every positive integer t.

Many techniques have been applied to the existence problem, and one particular approach, which eventually led to the study of orthogonal designs, was introduced by Williamson [25] in 1944. Williamson produced the following theorem:

THEOREM 1.2 (Williamson). *If A, B, C, and D are commuting, symmetric $n \times n$ matrices with entries ± 1 such that $A^2 + B^2 + C^2 + D^2 = 4nI_n$, then the array*

$$\begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix}$$

is a Hadamard matrix of order $4n$.

This array, which is called a *Williamson array*, was used to obtain Hadamard matrices of orders 148 and 172. Later, Baumert and Hall [1] produced a 12×12 Williamson array which gave an Hadamard matrix of order 156.

Because these arrays were such powerful tools for producing new Hadamard matrices, a great deal of work was done on finding such arrays, and in [7] Geramita, Geramita and Wallis generalized this idea with the following definition:

DEFINITION 1.3. An *orthogonal design of order* n *and type* $(u_1, \ldots, u_s)$, $u_i > 0$, on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ which satisfies

$$AA^t = \sum_{i=1}^{s} \left(u_i x_i^2\right) I_n .$$

The Williamson array given in Theorem 1.2 may be thought of as an orthogonal design of order 4 and type $(1, 1, 1, 1)$.

Geramita, Geramita and Wallis gave many constructions for orthogonal designs, and in 1976 Wallis [23], by using orthogonal designs, was able to reduce the existence problem for Hadamard matrices to a "finite problem" with the following theorem:

THEOREM 1.4 (Wallis). *Given any integer* q, *there exists* t, *dependent on* q, *such that an Hadamard matrix exists for every order* $2^s q$ *for* $s \geq t$.

Since orthogonal designs were first defined, there has been a great deal of research done on the existence and non-existence problems for orthogonal designs.

By considering an orthogonal design as a family of rational matrices, Geramita, Pullman, Shapiro and Wolfe (see [8], [19], [26], [27]) have produced many necessary conditions for the existence of orthogonal designs.

These results, when combined with a theorem of Geramita and Verner [9], were, at one time, thought to give sufficient conditions for the existence of orthogonal designs.

In Chapter 2, we completely solve the existence (and non-existence) problem for orthogonal designs of order 16. These results, together with those of Chapter 3, show that the algebraic conditions for the existence of orthogonal designs come nowhere near giving the complete answer for the non-existence problem of orthogonal designs and that there is still a great deal of work to be done in this area.

In Chapter 4 we are interested in amicable orthogonal designs. Wolfe [27] gives both algebraic and combinatorial non-existence proofs for amicable orthogonal designs, and in Chapter 4 we give further non-existence results.

In Chapter 5 we produce new types of designs which are extremely useful for constructing orthogonal designs, and, as examples of their use, we construct several orthogonal designs of orders 32, 64 and 128, and an infinite family in powers of 2.

Many other techniques have been used to construct orthogonal designs and Hadamard matrices. One very useful method, which is an extension of Theorem 1.2, was given by Goethals and Seidel in [11] and we state their result in the following theorem.

THEOREM 1.5 (Goethals and Seidel). *Suppose there exists four circulant matrices* A, B, C *and* D *of order* n *satisfying*

$$AA^t + BB^t + CC^t + DD^t = fI_n . \qquad (*)$$

Let R *be the back-diagonal matrix. Then*

$$GS = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

is an orthogonal design of order $4n$ *and type* $(u_1, u_2, \ldots, u_s)$ *on* $x_1, x_2, \ldots, x_s$ *when*

$$f = \sum_{j=1}^{s} u_j x_j^2 .$$

Further, GS *is skew or skew-type if* A *is skew or skew-type.*

The problem here is to find circulant matrices satisfying (*). Geramita, Wallis, Eades and Hain ([10], [3], [4], [5], [6]) have produced many results in this area.

In Chapter 6, we consider sequences introduced by Golay [12], [13], [14], Turyn [19], [20] and others which may be used to produce circulant matrices satisfying (*). These sequences are also of interest to engineers working on signal processing, for example, in radar and sonar.

NOTATION. Throughout this thesis, - denotes -1 and $\bar{x}$ denotes $-x$. We use I for the identity matrix.

APPENDIX.

Definitions.

Let $M = [m_{ij}]$ be a $m \times p$ matrix and $N = [n_{ij}]$ be a $n \times q$ matrix. The Kronecker product $M \times N$ is the $mn \times pq$ matrix whose (i,j) $n \times q$ block is $m_{ij}N$.

If $m = n$ and $p = q$ then the Hadamard product $M*N$ is the $m \times p$ matrix whose (i,j) entry is $m_{ij}n_{ij}$. The direct sum $M + N$ is the matrix given by

$$M + N = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} .$$

The sum of two matrices, $M + N$, is the matrix whose (i,j) entry is $m_{ij} + n_{ij}$.

A monomial matrix, P, is a matrix with entries taken from $\{0,\pm 1\}$ which satisfies

$$PP^t = I .$$

We now define an equivalence relation for orthogonal designs of the same order and type in terms of the following operations on an orthogonal design A :

(i) pre- and post- multiplying by monomial matrices ;

(ii) negating a variable ; and

(iii) interchanging the names of variables .

We say two orthogonal designs are equivalent if they are in the same equivalence class.

An orthogonal design of order n and type $(u_1,\ldots,u_s)$ is full if

$$\Sigma u_i = n .$$

That is, the design contains no zeros .

Algebraic theory of orthogonal designs.

Definition. A rational family of order n and type $[u_1,\ldots,u_s]$, u_i rational, is a collection of s rational matrices satisfying :

(i) $A_iA_i^t = u_iI$; and

(ii) $A_iA_j^t = - A_jA_i^t$, $i \neq j$.

It is obvious that the non-existence of a rational family of order n and type $[u_1,\ldots,u_s]$, with u_i integers , implies the non-existence of an orthogonal design of order n and type $(u_1,\ldots,u_s)$.

Geramita,Pullman,Shapiro and Wolfe have given necessary and sufficient conditions for the existence of such a rational family by appealing to the theory of Quadratic forms and Clifford algebras. See [8] ,[19] , [26] , [27] , and [28] .

Applications.

The most important use of orthogonal designs has been in the construction of Hadamard matrices (see [23]). Although orthogonal designs have yet to find practical applications , Hadamard matrices have been extensively used.

One important use of Hadamard matrices is in the construction of error correcting codes.

An error correcting code is a collection of code words with the property that each code word can still be recognised after a certain amount of error has been introduced into the system .

Hadamard matrices can be used to construct these codes in the following way.

Take an alphabet of 2n letters and assign each of the first n letters to different rows of an $n \times n$ Hadamard matrix H , and each of the second n letters to different rows of -H .

The rows become the code words for the alphabet , and the system is error correcting because if less than n/2 signs are changed in a code word , the origional code word can still be recognised.

CHAPTER 2

ORTHOGONAL DESIGNS OF ORDER 16

This chapter considers the existence problem for orthogonal designs of order 16. We construct several designs and show that any designs which cannot be obtained from these designs, or those previously known, do not exist.

2.1 Introduction

For convenience, we restate the following definition:

DEFINITION 2.1.1. *An orthogonal design of order n and type* $(u_1, u_2, \ldots, u_s)$, $u_i > 0$, on commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ which satisfies

$$AA^t = \sum_{i=1}^{s} u_i x_i^2 I_n$$

Alternatively, the rows of A are formally orthogonal and each row has precisely u_i entries of the type $\pm x_i$.

A may be considered as a matrix with entries in the field of quotients of the integral domain $Z[x_1, \ldots, x_s]$. Thus we obtain

$$A^t A = \sum_{i=1}^{s} u_i x_i^2 I_n$$

and so our alternative description applies equally well to the columns of A.

In [7] the following result was given:

THEOREM 2.1.2. *Let A be an orthogonal design of order n and type $(u_1, \ldots, u_s)$ on the variables $x_1, \ldots, x_s$, where $n = 2^a b$ with b odd, and $a = 4c + d$ where $0 \le d < 4$. If $\rho(n) = 8c + 2^d$ is Radon's* *function, then* $s \leq \rho(n)$.

Proof. Write $A = A_1x_1 + \dots + A_sx_s$ where A_i are $(0, 1, -1)$ matrices of order n . Now

$$AA^t = \sum_{i=1}^{s} u_i x_i^2 I_n \iff A_i A_i^t = u_i I_n \ , \quad i = 1, \dots, s \ ,$$

and $A_i A_j^t + A_j A_i^t = 0$ for $i \neq j$. If we replace A_i by the real matrix $\frac{1}{\sqrt{u_i}} A_i = B_i$, then the B_i are real orthogonal matrices satisfying

$B_i B_j^t + B_j B_i^t = 0$ for $i \neq j$, and Radon [15½] has shown that there do not exist more than $\rho(n)$ such real matrices. This completes the proof.

We note that $\rho(16) = 9$, and therefore we can have, at most, 9 variables in an orthogonal design of order 16 . We also note that if A is an orthogonal design and P and Q are monomial matrices of the same order, then PAQ is an orthogonal design of the same type as A . This allows us to interchange and negate rows and columns of A without essentially changing the design.

In Section 2.2 we give the statement of the main theorem; in Sections 2.3 and 2.4 we give proofs for existence and non-existence respectively.

2.2 Main Result

THEOREM 2.2.1. *The following orthogonal designs, and those that can be obtained from them by equating variables or setting variables equal to zero, are the only orthogonal designs of order* 16 *:*

(1, 1, 1, 1, 1, 1, 1, 1, 1)	(1, 1, 1, 1, 1, 2, 2, 2, 2)
(1, 1, 1, 1, 1, 1, 1, 1, 2)	(1, 1, 1, 1, 1, 2, 3, 3, 3)
(1, 1, 1, 1, 1, 1, 1, 1, 4)	(1, 1, 1, 1, 2, 2, 2, 2, 2)
(1, 1, 1, 1, 1, 1, 1, 1, 8)	(1, 1, 2, 2, 2, 2, 2, 2, 2)
(1, 1, 1, 1, 1, 1, 2, 2, 2)	
(1, 1, 1, 1, 1, 1, 3, 3)	(1, 1, 1, 1, 2, 2, 3, 3)
(1, 1, 1, 1, 1, 1, 4, 4)	(1, 1, 1, 1, 2, 2, 4, 4)
(1, 1, 1, 1, 1, 1, 5, 5)	(1, 1, 2, 2, 2, 2, 3, 3) .

The proof of this theorem is given in two parts: first we establish the existence of the designs quoted in the theorem then we verify non-existence of all the other designs.

2.3 Existence of Orthogonal Designs of Order 16

We consider the following matrix of order 16 ;

$$R = \left[\begin{array}{cccc|cccc|cccc|cccc}
X_1 & X_2 & X_3 & M & X_4 & N & Q & P & A & B & C & D & E & F & G & H \\
-X_2 & X_1 & M & -X_3 & N & -X_4 & P & -Q & B & -A & D & -C & F & -E & H & -G \\
-X_3 & -M & X_1 & X_2 & Q & -P & -X_4 & N & C & -D & -A & B & G & -H & -E & F \\
-M & X_3 & -X_2 & X_1 & -P & -Q & N & X_4 & -D & -C & B & A & -H & -G & F & E \\
\hline
-X_4 & -N & -Q & P & X_1 & X_2 & X_3 & -M & E & -F & -G & H & -A & B & C & -D \\
-N & X_4 & P & Q & -X_2 & X_1 & -M & -X_3 & -F & -E & H & G & B & A & -D & -C \\
-Q & -P & X_4 & -N & -X_3 & M & X_1 & X_2 & -G & -H & -E & -F & C & D & A & B \\
-P & Q & -N & -X_4 & M & X_3 & -X_2 & X_1 & -H & G & -F & E & D & -C & B & -A \\
\hline
-A & -B & -C & D & -E & F & G & H & X_1 & X_2 & X_3 & R & X_4 & S & T & U \\
-B & A & D & C & F & E & H & -G & -X_2 & X_1 & R & -X_3 & S & -X_4 & U & -T \\
-C & -D & A & -B & G & -H & E & F & -X_3 & -R & X_1 & X_2 & T & -U & -X_4 & S \\
-D & C & -B & -A & -H & -G & F & -E & -R & X_3 & -X_2 & X_1 & -U & -T & S & X_4 \\
\hline
-E & -F & -G & H & A & -B & -C & -D & -X_4 & -S & -T & U & X_1 & X_2 & X_3 & -R \\
-F & E & H & G & -B & -A & -D & C & -S & X_4 & U & T & -X_2 & X_1 & -R & -X_3 \\
-G & -H & E & -F & -C & D & -A & -B & -T & -U & X_4 & -S & -X_3 & R & X_1 & X_2 \\
-H & G & -F & -E & D & C & -B & A & -U & T & -S & -X_4 & R & X_3 & -X_2 & X_1
\end{array}\right] .$$

Throughout this matrix 'X' should read as 'x'.

This matrix is an orthogonal design if and only if the following conditions are satisfied:

$$\{|M|,\ |N|,\ |Q|,\ |P|\} = \{|R|,\ |S|,\ |T|,\ |U|\}$$

$$\begin{aligned}
(M+R)D + (N+S)F + (Q+T)G + (P+U)H &= 0\\
(M+R)C + (N+S)E + (Q-T)H - (P-U)G &= 0\\
(M+R)B + (N-S)H - (Q+T)E - (P-U)F &= 0\\
(M+R)A + (N-S)G - (Q-T)F + (P+U)E &= 0\\
(R-M)H + (N+S)B + (Q+T)C + (P-U)D &= 0\\
(R-M)G + (N+S)A + (Q-T)D - (P+U)C &= 0\\
(R-M)F + (N-S)D - (Q+T)A - (P+U)B &= 0\\
(R-M)E + (N-S)C - (Q-T)B + (P-U)A &= 0
\end{aligned} \tag{a}$$

$$AH - BG + CF - DE = 0\ .$$

The following designs can be obtained by choosing the variables of R in the manner indicated:

(1, 1, 1, 1, 1, 1, 1, 1, 4) $M = S = x_5$, $N = T = x_6$,
$R = -Q = x_7$, $P = U = x_8$,
$A = D = G = -F = x_9$,
all others zero;

(1, 1, 1, 1, 1, 1, 1, 1, 8) $M = S = x_5$, $N = T = x_6$,
$R = P = x_7$, $Q = U = x_8$,
$A = -B = D = C = G = -H = -E = -F = x_9$;

(1, 1, 1, 1, 1, 1, 2, 2, 2) $A = x_5$, $P = U = x_6$, $R = -D = -M = x_7$,
$N = -S = F = x_8$, $Q = -T = G = x_9$,
all others zero;

(1, 1, 1, 1, 1, 1, 4, 4) $M = S = x_5$, $R = -Q = x_6$,
$A = G = D = -F = x_7$, $B = H = C = -E = x_8$,
all others zero;

(1, 1, 1, 1, 1, 1, 5, 5) $M = S = x_5$, $N = U = x_6$,
$A = G = P = R = C = -E = x_7$,
$B = H = D = -F = -Q = -T = x_8$;

DEFINITION. 2.4.0. The Hilbert norm-residue symbol $(a,b)_p$ is defined to be $+1$ or -1, according as the congruence

$$ax^2+by^2 \equiv z^2 \pmod{p^m}$$

does or does not have a solution in integers x,y,z, not all multiples of p, for arbitrarily high powers of p.

$(1, 1, 1, 1, 2, 2, 4, 4)$ $\quad D = F = x_5 \,, \quad G = H = x_6 \,,$

$$M = R = E = C = -N = -S = x_7 \,,$$

$$Q = T = B = A = -P = -U = x_8 \,.$$

The orthogonal design of order 16 and type $(1, 1, 1, 1, 2, 2, 2, 2, 2)$ is:

$$\left[\begin{array}{cccccccc|cccccccc}
x_5 & x_5 & x_1 & x_3 & x_2 & x_4 & 0 & 0 & x_6 & x_7 & x_8 & x_9 & x_7 & -x_6 & x_9 & -x_8 \\
-x_5 & x_5 & x_3 & -x_1 & x_4 & -x_2 & 0 & 0 & x_7 & -x_6 & -x_9 & x_8 & -x_6 & -x_7 & x_8 & x_9 \\
-x_1 & -x_3 & x_5 & x_5 & 0 & 0 & -x_2 & x_4 & x_9 & x_8 & -x_6 & x_7 & -x_8 & x_9 & -x_7 & -x_6 \\
-x_3 & x_1 & -x_5 & x_5 & 0 & 0 & x_4 & x_2 & -x_8 & x_9 & x_7 & x_6 & -x_9 & -x_8 & -x_6 & x_7 \\
-x_2 & -x_4 & 0 & 0 & x_5 & x_5 & x_1 & -x_3 & x_7 & x_6 & x_8 & -x_9 & -x_6 & x_7 & x_9 & x_8 \\
-x_4 & x_2 & 0 & 0 & -x_5 & x_5 & -x_3 & -x_1 & x_6 & -x_7 & x_9 & x_8 & x_7 & x_6 & -x_8 & x_9 \\
0 & 0 & x_2 & -x_4 & -x_1 & x_3 & x_5 & x_5 & -x_9 & x_8 & -x_7 & x_6 & x_8 & x_9 & x_6 & x_7 \\
0 & 0 & -x_4 & -x_2 & x_3 & x_1 & -x_5 & x_5 & -x_8 & -x_9 & x_6 & x_7 & -x_9 & x_8 & x_7 & -x_6 \\
\hline
-x_6 & -x_7 & -x_8 & -x_9 & -x_7 & -x_6 & -x_8 & x_9 & x_5 & x_5 & x_1 & 0 & x_2 & -x_4 & x_3 & 0 \\
-x_7 & x_6 & x_9 & -x_8 & -x_6 & x_7 & -x_9 & -x_8 & -x_5 & x_5 & 0 & -x_1 & -x_4 & -x_2 & 0 & -x_3 \\
-x_9 & -x_8 & x_6 & -x_7 & x_9 & -x_8 & x_7 & -x_6 & -x_1 & 0 & x_5 & x_5 & x_3 & 0 & -x_2 & -x_4 \\
x_8 & -x_9 & -x_7 & -x_6 & x_8 & x_9 & -x_6 & -x_7 & 0 & x_1 & -x_5 & x_5 & 0 & -x_3 & -x_4 & x_2 \\
-x_7 & x_6 & -x_9 & x_8 & x_6 & -x_7 & -x_9 & -x_8 & -x_2 & x_4 & -x_3 & 0 & x_5 & x_5 & x_1 & 0 \\
x_6 & x_7 & -x_8 & -x_9 & -x_7 & -x_6 & x_8 & -x_9 & x_4 & x_2 & 0 & x_3 & -x_5 & x_5 & 0 & -x_1 \\
x_8 & -x_9 & x_7 & x_6 & -x_8 & -x_9 & -x_6 & -x_7 & -x_3 & 0 & x_2 & x_4 & -x_1 & 0 & x_5 & x_5 \\
x_9 & x_8 & x_6 & -x_7 & x_9 & -x_8 & -x_7 & x_6 & 0 & x_3 & x_4 & -x_2 & 0 & x_1 & -x_5 & x_5
\end{array}\right].$$

The remaining designs are given by Geramita and Wallis [10] ~~or~~ and may also be obtained from product designs.

← insert

2.4 Non-Existence of Orthogonal Designs of Order 16

The following theorems by D. Shapiro [19] and Geramita and Verner [9] give strong non-existence results.

THEOREM 2.4.1 (Shapiro). *If* $n \equiv 16 \pmod{32}$ *, then there exists an orthogonal design of type* $(a_1, a_2, \ldots, a_9)$ *only if the Hasse invariant*

$s_p(a_1, \ldots, a_9)$ equals 1 at every prime p.

We note that

$$s_p(a_1, \ldots, a_t) = \prod_{1 \le i < j \le t} (a_i, a_j)_p$$

where $(a_i, a_j)_p$ is the Hilbert norm residue symbol (see [15]).

THEOREM 2.4.2 (Geramita and Verner). *If there exists an orthogonal design of type $(u_1, u_2, \ldots, u_s)$ in order $n \equiv 0 \pmod 4$ and $\sum_{i=1}^{s} u_i = n - 1$ then there exists an orthogonal design of type $(1, u_1, u_2, \ldots, u_s)$ in order n.*

These theorems are used to show the following 9-tuples and 8-tuples cannot be the type of an orthogonal design:

(1, 1, 1, 1, 1, 1, 1, 1, 7)	(1, 1, 1, 1, 1, 1, 3, 3, 3)
(1, 1, 1, 1, 1, 1, 1, 2, 3)	(1, 1, 1, 1, 1, 1, 3, 3, 4)
(1, 1, 1, 1, 1, 1, 1, 2, 5)	(1, 1, 1, 1, 1, 2, 2, 2, 3)
(1, 1, 1, 1, 1, 1, 1, 2, 6)	(1, 1, 1, 1, 1, 2, 2, 2, 4)
(1, 1, 1, 1, 1, 1, 1, 3, 3)	(1, 1, 1, 1, 1, 2, 2, 2, 5)
(1, 1, 1, 1, 1, 1, 1, 3, 5)	(1, 1, 1, 1, 1, 2, 2, 3, 3)
(1, 1, 1, 1, 1, 1, 1, 4, 4)	(1, 1, 1, 1, 2, 2, 2, 2, 3)
(1, 1, 1, 1, 1, 1, 2, 2, 5)	(1, 1, 1, 1, 2, 2, 2, 3, 3)
(1, 1, 1, 1, 1, 1, 2, 3, 3)	(1, 1, 1, 2, 2, 2, 2, 2, 2)
(1, 1, 1, 1, 1, 1, 2, 3, 4)	(1, 1, 1, 2, 2, 2, 2, 2, 3)
(1, 1, 1, 1, 1, 3, 3, 4)	(1, 1, 1, 2, 2, 2, 3, 3)
(1, 1, 1, 1, 2, 2, 2, 5)	(1, 1, 2, 2, 2, 2, 2, 3) .

To complete the proof of the main theorem we need only show that no designs of the following types can exist in order 16 :

$$
\begin{array}{ll}
(1, 1, 1, 1, 1, 4, 7) & (1) \\
(1, 1, 1, 1, 1, 2, 7) & (2) \\
(1, 1, 1, 1, 1, 1, 7) & (3) \\
(1, 1, 1, 1, 1, 1, 2, 3) & (4) \\
(1, 1, 1, 2, 2, 2, 7) & (5) \\
(1, 1, 1, 1, 2, 2, 3, 5) & (6) .
\end{array}
\qquad (b)
$$

We begin by considering (1) to (4) of (b).

Let A be a design of type $(1, 1, 1, 1, 1, s_1, \ldots, s_k)$, $k \geq 4$, in order 16.

We can force the diagonal blocks of A to be of the form

$$\begin{bmatrix} x_1 & x_2 & x_3 & a_i \\ \bar{x}_2 & x_1 & a_i & \bar{x}_3 \\ \bar{x}_3 & \bar{a}_i & x_1 & x_2 \\ \bar{a}_i & x_3 & \bar{x}_2 & x_1 \end{bmatrix}$$

for some a_i's.

If the variable x_4 does not appear in any diagonal block, we can assume that A is the matrix R.

Now we consider the following matrices:

$$X_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&-&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&- \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&-&0 \\ 0&0&0&0&0&0&0&- \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&-&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&-&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 0&-&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&-&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{bmatrix}.$$

Let

$$Y_i = \begin{bmatrix} X_i & 0 \\ 0 & X_i \end{bmatrix}, \quad i = 1, \ldots, 4.$$

Now, $Y_1 R Y_1^t$ is similar to R but with the variables x_2 and x_3 interchanged. $(MNQP)$ is changed to $(\overline{M}Q N\overline{P})$ and $(RSTU)$ to $(\overline{R}TS\overline{U})$.

$Y_2 R Y_2^t$ is similar to R but with the variables x_3 and x_4 interchanged. $(MNQP)$ is changed to $(NM\overline{Q}\overline{P})$ and $(RSTU)$ to $(SR\overline{T}\overline{U})$.

$Y_3 R Y_4$ is also similar to R but with x_1 replaced by x_2 and x_2 replaced by $-x_1$. $(MNQP)$ is changed to $(MNP\overline{Q})$ and $(RSTU)$ to $(RSU\overline{T})$.

Hence we may assume

$$x_5 = M = R \ , \quad x_5 = M = S \ , \text{ or } \quad x_5 = A \ .$$

Simple permutations and negations of rows and columns 9 to 16 ensure that all the above signs are positive.

If the variable x_4 appears in all the diagonal blocks, we can permute rows and columns until x_5 appears in position (1, 5) and (9, 13) . Then, on interchanging the names of the variables x_4 and x_5 , we obtain R with $x_5 = M = R$.

If the variable x_4 appears in only two diagonal 4×4 blocks, it can be seen that the variable x_5 cannot appear in the top right hand 8×8 block. If the variable x_5 does not appear in any diagonal 4×4 block, we can see that this is equivalent to R with $M = S = x_5$.

If the variables x_4 and x_5 both appear in two diagonal 4×4 blocks, we interchange rows and columns 1 and 2 , 5 and 6 , 9 and 10 , 11 and 13 , 12 and 14 , and 13 and 14 , then change the signs of rows and columns 1, 5, 7, 8, 9 and 11 . We obtain a matrix similar to R with $M = S = x_5$, but with x_3 replaced by $-x_4$, x_4 replaced by $-x_5$, and x_5 replaced by x_3 .

Therefore, any orthogonal design of order 16 and type

$(1, 1, 1, 1, 1, s_1, \ldots, s_k)$, $k \leq 4$, can be written as R with $M = R = x_5$, $M = S = x_5$ or $A = x_5$.

Now we consider R with $M = R = x_5$.

The equations (a) give

$$A = B = C = D = 0$$

$$(N+S)F + (Q+T)G + (P+U)H = 0$$
$$(N+S)E + (Q-T)H - (P-U)G = 0$$
$$(N-S)H - (Q+T)E - (P-U)F = 0$$
$$(N-S)G - (Q-T)F + (P+U)E = 0 .$$

By trying all possibilities, we can see that these equations yield no designs of the types given in (b).

Similarly we can show that no designs in (b) can be obtained from R with $M = S = x_5$ or $A = x_5$.

Hence, we have shown that there are no designs of the types given in (1) to (4) of (b).

We now assume that there exists a design, B , of order 16 and type (1, 1, 1, 1, 2, 2, 3, 5) .

Since this is a design which contains no zeros and has four variables which appear once per row and column, we see that B = R (given above) or B is the following matrix:

$$Q = \left[\begin{array}{cccc|cccc|cccc|cccc}
X_1 & X_2 & X_3 & M & X_4 & N & Q & P & A & B & C & D & E & F & G & H \\
-X_2 & X_1 & M & -X_3 & N & -X_4 & P & -Q & B & -A & D & -C & F & -E & H & -G \\
-X_3 & -M & X_1 & X_2 & Q & -P & -X_4 & N & C & -D & -A & B & G & -H & -E & F \\
-M & X_3 & -X_2 & X_1 & -P & -Q & N & X_4 & -D & -C & B & A & -H & -G & F & E \\
\hline
-X_4 & -N & -Q & P & X_1 & X_2 & X_3 & -M & D & C & -B & -A & -H & -G & F & E \\
-N & X_4 & P & Q & -X_2 & X_1 & -M & -X_3 & C & -D & -A & B & -G & H & E & -F \\
-Q & -P & X_4 & -N & -X_3 & M & X_1 & X_2 & -B & A & -D & C & F & -E & H & -G \\
-P & Q & -N & -X_4 & M & X_3 & -X_2 & X_1 & A & B & C & D & -E & -F & -G & -H \\
\hline
-A & -B & -C & D & -D & -C & B & -A & X_1 & X_2 & X_3 & X_4 & R & S & T & U \\
-B & A & D & C & -C & D & -A & -B & -X_2 & X_1 & X_4 & -X_3 & S & -R & U & -T \\
-C & -D & A & -B & B & A & D & -C & -X_3 & -X_4 & X_1 & X_2 & T & -U & -R & S \\
-D & C & -B & -A & A & -B & -C & -D & -X_4 & X_3 & -X_2 & X_1 & -U & -T & S & R \\
\hline
-E & -F & -G & H & H & G & -F & E & -R & -S & -T & U & X_1 & X_2 & X_3 & -X_4 \\
-F & E & H & G & G & -H & E & F & -S & R & U & T & -X_2 & X_1 & -X_4 & -X_3 \\
-G & -H & E & -F & -F & -E & -H & G & -T & -U & R & -S & -X_3 & X_4 & X_1 & X_2 \\
-H & G & -F & -E & -E & F & G & H & -U & T & -S & -R & X_4 & X_3 & -X_2 & X_1
\end{array}\right].$$

Throughout this matrix 'X' should read as 'x'

Equations (a) cannot yield a design of type (1, 1, 1, 1, 2, 2, 3, 5) and therefore $B \neq R$.

The matrix Q is an orthogonal design only if the following equations are satisfied:

$$A^2 + B^2 + C^2 + D^2 - E^2 - F^2 - G^2 - H^2 = 0$$

$$MD - NC + QB - PA + RE + SF + TG + UH = 0$$

$$MC + ND - QA - PB + SE - RF + UG - TH = 0$$

$$-MB + NA + QD - PC + TE - UF - RG + SH = 0$$

$$-MA - NB - QC - PD - UE - FT + SG + RH = 0 .$$

By considering these equations, it can be seen that Q cannot produce a (1, 1, 1, 1, 2, 2, 3, 5) design. Therefore, there is no orthogonal design of order 16 and type (1, 1, 1, 1, 2, 2, 3, 5) .

We now assume that there is an orthogonal design, C , of order 16 and type (1, 1, 1, 2, 2, 2, 7) .

As before, we force the diagonal 4×4 blocks of C to be

$$\begin{bmatrix} x_1 & x_2 & x_3 & a_i \\ \overline{x}_2 & x_1 & a_i & \overline{x}_3 \\ \overline{x}_3 & \overline{a}_i & x_1 & x_2 \\ \overline{a}_i & x_3 & \overline{x}_2 & x_1 \end{bmatrix}$$

for some a_i's .

Since, in C , there are three variables which each appear twice per row and column, at least one of these variables does not appear in any diagonal 4×4 block. Let this variable be x_4 .

By using various permutations and negations of rows and columns, we may assume x_4 appears in position (1, 9) of C and $\pm x_4$ appears in position (5, 13) of C .

The variable x_4 appears again in row 1 and is either in a different 4×4 block from the first x_4 , or is in the same block. If the first possibility is true then C may be written in the following way:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
X_1 & X_2 & X_3 & M & A & B & C & D & X_4 & E & F & G & X_4 & P & Q & R \\
-X_2 & X_1 & M & -X_3 & B & -A & D & -C & E & -X_4 & G & -F & P & -X_4 & R & -Q \\
-X_3 & -M & X_1 & X_2 & C & -D & -A & B & F & -G & -X_4 & E & Q & -R & -X_4 & P \\
-M & X_3 & -X_2 & X_1 & -D & -C & B & A & -G & -F & E & X_4 & -R & -Q & P & X_4 \\
\hline
-A & -B & -C & D & X_1 & X_2 & X_3 & N & X_4 & p & q & r & -X_4 & e & f & g \\
-B & A & D & C & -X_2 & X_1 & N & -X_3 & p & -X_4 & r & -q & e & X_4 & g & -f \\
-C & -D & A & -B & -X_3 & -N & X_1 & X_2 & q & -r & -X_4 & p & f & -g & X_4 & e \\
-D & C & -B & -A & -N & X_3 & -X_2 & X_1 & -r & -q & p & X_4 & -g & -f & e & -X_4 \\
\hline
-X_4 & -E & -F & G & -X_4 & -p & -q & r & X_1 & X_2 & X_3 & m & A & -B & -C & d \\
-E & X_4 & G & F & -p & X_4 & r & q & -X_2 & X_1 & m & -X_3 & -B & -A & d & C \\
-F & -G & X_4 & -E & -q & -r & X_4 & -p & -X_3 & -m & X_1 & X_2 & -C & -d & -A & -B \\
-G & F & -E & -X_4 & -r & q & -p & -X_4 & -m & X_3 & -X_2 & X_1 & -d & C & -B & A \\
\hline
-X_4 & -P & -Q & R & X_4 & -e & -f & g & -A & B & C & d & X_1 & X_2 & X_3 & n \\
-P & X_4 & R & Q & -e & -X_4 & g & f & B & A & d & -C & -X_2 & X_1 & n & -X_3 \\
-Q & -R & X_4 & -P & -f & -g & -X_4 & -e & C & -d & A & B & -X_3 & -n & X_1 & X_2 \\
-R & Q & -P & -X_4 & -g & f & -e & X_4 & -d & -C & B & -A & -n & X_3 & -X_2 & X_1
\end{array}\right] . \qquad \text{(I)}$$

Throughout this matrix 'X' should read as 'x'.

If the second possibility is true then let

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & - & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

We put

$$Z_1 = \begin{bmatrix} X & & & \\ & X & & 0 \\ 0 & & Y & \\ & & & Y \end{bmatrix} , \quad Z_2 = \begin{bmatrix} Y & & & \\ & Y & & 0 \\ 0 & & X & \\ & & & X \end{bmatrix} ,$$

$$Z_3 = \begin{bmatrix} X & & & \\ & X & & 0 \\ 0 & & Y & \\ & & & \overline{X} \end{bmatrix} , \quad Z_4 = \begin{bmatrix} Y & & & \\ & Y & & \\ & & X & \\ & & & \overline{Y} \end{bmatrix}$$

and consider the following operations on C : Z_1CZ_2 , $Z_1CZ_1^t$, Z_3CZ_4 , changing the signs of rows and columns 2, 3, 6, 7, 10, 11, 14 and 15 , interchanging rows and columns 13 and 15 , and 14 and 16 , and changing the signs of rows and columns 15 and 16 .

By using these operations where necessary, and in some cases relabelling the variables, we can see that C may be written in one of the following two ways:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
X_1 & X_2 & X_3 & M & A & B & C & D & X_4 & X_4 & P & Q & E & F & G & H \\
-X_2 & X_1 & M & -X_3 & B & -A & D & -C & X_4 & -X_4 & Q & -P & F & -E & H & -G \\
-X_3 & -M & X_1 & X_2 & C & -D & -A & B & P & -Q & -X_4 & X_4 & G & -H & -E & F \\
-M & X_3 & -X_2 & X_1 & -D & -C & B & A & -Q & -P & X_4 & X_4 & -H & -G & F & E \\
\hline
-A & -B & -C & D & X_1 & X_2 & X_3 & N & e & f & g & h & X_4 & R & X_4 & S \\
-B & A & D & C & -X_2 & X_1 & N & -X_3 & f & -e & h & -g & R & -X_4 & S & -X_4 \\
-C & -D & A & -B & -X_3 & -N & X_1 & X_2 & g & -h & -e & f & X_4 & -S & -X_4 & R \\
-D & C & -B & -A & -N & X_3 & -X_2 & X_1 & -h & -g & f & e & -S & -X_4 & R & X_4 \\
\hline
-X_4 & -X_4 & -P & Q & -e & -f & -g & h & X_1 & X_2 & X_3 & -M & a & b & c & d \\
-X_4 & X_4 & Q & P & -f & e & h & g & -X_2 & X_1 & -M & -X_3 & b & -a & d & -c \\
-P & -Q & X_4 & -X_4 & -g & -h & e & -f & -X_3 & M & X_1 & X_2 & c & -d & -a & b \\
-Q & P & -X_4 & -X_4 & -h & g & -f & -e & M & X_3 & -X_2 & X_1 & -d & -c & b & a \\
\hline
-E & -F & -G & H & -X_4 & -R & -X_4 & S & -a & -b & -c & d & X_1 & X_2 & X_3 & -N \\
-F & E & H & G & -R & X_4 & S & X_4 & -b & a & d & c & -X_2 & X_1 & -N & X_3 \\
-G & -H & E & -F & -X_4 & -S & X_4 & -R & -c & -d & a & -b & -X_3 & N & X_1 & X_2 \\
-H & G & -F & -E & -S & X_4 & -R & -X_4 & -d & c & -b & -a & N & X_3 & -X_2 & X_1
\end{array}\right], \qquad \text{(II)}$$

Throughout this matrix 'X' should read as 'x'.

or

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
X_1 & X_2 & X_3 & M & A & B & C & D & X_4 & X_4 & P & Q & E & F & G & H \\
-X_2 & X_1 & M & -X_3 & B & -A & D & -C & X_4 & -X_4 & Q & -P & F & -E & H & -G \\
-X_3 & -M & X_1 & X_2 & C & -D & -A & B & P & -Q & -X_4 & X_4 & G & -H & -E & F \\
-M & X_3 & -X_2 & X_1 & -D & -C & B & A & -Q & -P & X_4 & X_4 & -H & -G & F & E \\
\hline
-A & -B & -C & D & X_1 & X_2 & X_3 & N & -F & -E & G & H & X_4 & X_4 & R & S \\
-B & A & D & C & -X_2 & X_1 & N & -X_3 & -E & F & H & -G & X_4 & -X_4 & S & -R \\
-C & -D & A & -B & -X_3 & -N & X_1 & X_2 & G & -H & F & -E & R & -S & -X_4 & X_4 \\
-D & C & -B & -A & -N & X_3 & -X_2 & X_1 & -H & -G & -E & -F & -S & -R & X_4 & X_4 \\
\hline
-X_4 & -X_4 & -P & Q & F & E & -G & H & X_1 & X_2 & X_3 & -M & -B & -A & C & -D \\
-X_4 & X_4 & Q & P & E & -F & H & G & -X_2 & X_1 & -M & -X_3 & -A & B & -D & -C \\
-P & -Q & X_4 & -X_4 & -G & -H & -F & E & -X_3 & M & X_1 & X_2 & C & D & B & -A \\
-Q & P & -X_4 & -X_4 & -H & G & E & F & M & X_3 & -X_2 & X_1 & D & -C & -A & -B \\
\hline
-E & -F & -G & H & -X_4 & -X_4 & -R & S & B & A & -C & -D & X_1 & X_2 & X_3 & -N \\
-F & E & H & G & -X_4 & X_4 & S & R & A & -B & -D & C & -X_2 & X_1 & -N & -X_3 \\
-G & -H & E & -F & -R & -S & X_4 & -X_4 & -C & D & -B & A & -X_3 & N & X_1 & X_2 \\
-H & G & -F & -E & -S & R & -X_4 & -X_4 & D & C & A & B & N & X_3 & -X_2 & X_1
\end{array}\right]. \qquad \text{(III)}$$

Throughout this matrix 'X' should read as 'x'.

The matrix (I) gives an orthogonal design only if the following equations are satisfied:

$$M = -d = -N \ , \quad m = -n = -D$$

$$pE + Pe + Fq + Qf + rG + Rg = 0$$

$$p + e - E + P = 0$$

$$q + f + Q - F = 0$$

$$-r - g + G - R = 0$$

$$rF - fR - Gq + gQ = 0$$

$$-Er + eR + pG - gP = 0$$

$$-Eq + eQ + pF - Pf = 0$$

$$M(G-R) + m(G-r) - B(p+P) - C(q+Q) = 0$$

$$-M(E-P) - m(E-p) - A(q+Q) - B(r+R) = 0$$

$$M(R-G) - m(R+g) + B(E-e) + C(F-f) = 0$$

$$M(Q-F) - m(Q+f) - C(G-g) + A(E-e) = 0$$

$$-M(P-E) + m(P-e) + A(F-f) + B(G-g) = 0$$

$$A(R-r) - B(Q-q) + C(P-p) = 0$$
$$-A(G+g) + B(F+f) - C(E+e) = 0 \ .$$

By tedious and systematic elimination we can see that (I) cannot produce a (1, 1, 1, 2, 2, 2, 7) design.

The matrix (II) gives an orthogonal design only if the following equations are satisfied:

$$-Ah + aH + Bg + Gb - Cf - cF - De - dE = 0$$
$$e + f + E + G = 0$$
$$-e + f - F - H = 0$$
$$g - h + E - G = 0$$
$$-g - h - F + H = 0$$
$$-A - C - a - b = 0$$
$$B + D - b + a = 0$$
$$-A + C - c - d = 0$$
$$B - D - d + c = 0 \ .$$

By systematic elimination we can see that (II) cannot produce a (1, 1, 1, 2, 2, 2, 7) design.

The matrix (III) gives an orthogonal design only if the following equations are satisfied:

$$D(M+N) + G(P+R) + H(Q+S) = 0$$
$$C(M+N) + H(P-R) - G(Q-S) = 0$$
$$-B(M+N) + F(P-S) - E(Q-R) = 0$$
$$-A(M+N) - E(P+S) - F(Q+R) = 0$$
$$H(M-N) - C(P+R) - D(Q-S) = 0$$
$$G(M-N) - D(P-R) + C(Q+S) = 0$$
$$-F(M-N) - B(P+S) + A(Q-R) = 0$$
$$-E(M-N) + A(P-S) + B(Q+R) = 0$$

$$-A(G+H) + B(G-H) + C(E-F) + D(E+F) = 0 \ .$$

Again we cannot obtain a (1, 1, 1, 2, 2, 2, 7) design.

Therefore, there is no (1, 1, 1, 2, 2, 2, 7) design of order 16 .

This completes the proof of the main theorem.

In the next chapter we extend some of the ideas used here to obtain a strong non-existence proof for higher orders.

CHAPTER 3

A STRONG NON-EXISTENCE THEOREM

It had previously been conjectured, because of the algebraic theory, that: any k-tuple, where $k \leq \rho(n) - l$ and l is "about" 6, was the type of an orthogonal design of order n. Hence, the theorem of this chapter is surprising because it shows that the combinatorial structure is much more restrictive than previously thought.

The theorem is:

THEOREM 3.1. *There is no orthogonal design of order* n, $n > 40$, *and type* $(1, 1, 1, 1, 1, n-5)$.

Proof. The proof appears in the paper [16] bound into the rear of this thesis.

Peter Eades [5], for example, has used this theorem to obtain restrictions on the generalization of the Goethals-Seidel array.

CHAPTER 4

AMICABLE ORTHOGONAL DESIGNS

In this chapter we will be mainly concerned with the non-existence problem for amicable orthogonal designs. We do, however, give some constructions for amicable orthogonal designs and, in the next chapter, we produce a very powerful tool for constructing orthogonal designs from amicable orthogonal designs.

4.1 Introduction

In [7] Geramita, Geramita, and Wallis give the following construction: If there is an orthogonal design of order n and type (s_1, s_2) then:

(i) there is an orthogonal design of order $2n$ and type (s_1, s_1, s_2, s_2) ; and

(ii) there is an orthogonal design of order $4n$ and type $(s_1, s_1, 2s_1, s_2, s_2, 2s_2)$.

This construction depends on the existence of orthogonal designs, A and B , of the same order which satisfy

$$AB^t = BA^t .$$

These designs, apart from being interesting in their own right, have become very useful for constructing orthogonal designs, and so it became necessary to give such designs a name.

DEFINITION 4.1.1. Let A and B be orthogonal designs of the same order and types $(a_1, \ldots, a_s)$ and $(b_1, \ldots, b_t)$ respectively. If

$$AB^t = BA^t$$

then we say A and B are *amicable orthogonal designs of types* $((a_1, \ldots, a_s); (b_1, \ldots, b_t))$.

Most constructions seem to produce very few orthogonal designs of order n and type $(a_1, \dots, a_k)$ with

$$\sum_{i=1}^{k} u_i = n$$

and some a_i odd, especially 1.

This therefore, leads us to consider amicable orthogonal designs A and B of order n and types $((a_1, \dots, a_s); (b_1, \dots, b_t))$ with $a_1 = 1$ and

$$\sum_{i=1}^{s} a_i = \sum_{i=1}^{t} b_i = n .$$

The restriction placed upon the designs by insisting $a_1 = 1$ is quite strong as we see in the following lemma.

LEMMA 4.1.2. *If A and B are amicable orthogonal designs of order n and types $((1, a_2, a_3, \dots, a_s); (b_1, b_2, \dots, b_t))$ then there exist monomial matrices P and Q such that*

$$PAQ = Ix_1 + X = A_1$$

and

$$PBQ = B_1$$

with

$$A_1B_1^t = B_1A_1^t , \quad X^t = -X , \text{ and } B_1^t = B_1 .$$

Proof. We find a pair of monomials, P and Q, which force x_1 to appear along the diagonal of A. The other results follow by considering the facts that A is an orthogonal design and

$$AB^t = BA^t .$$

We may therefore assume $A = Ix_1 + X$, where X is an orthogonal design of type $(a_2, \dots, a_s)$ with $X^t = -X$, and $B^t = B$.

In [17] we gave the solution to the problem of existence of such amicable orthogonal designs in order 8 . Since then it has been discovered that some of the non-existence results are examples of more general results.

4.2 Non-existence Results

Wolfe [28], by considering the algebraic existence problem, gives bounds on the number of variables in amicable orthogonal designs. For example, in order 8 , we have the following:

No. of variables in B	1	2	3	4	5	6	7	8
No. of variables in A	5	4	4	4	1	-	-	- .

This seems to be the only general algebraic restrictions in powers of two. It is certainly not the whole story of non-existence as Wolfe [27] shows in the following two theorems.

THEOREM 4.2.1 (Wolfe). *Suppose A and B are amicable orthogonal designs of order $n \equiv 0 \pmod 4$ where A is of type $(1, 1, 1, a_1, \ldots, a_s)$, $s \geq 0$, and B is of type $(b_1, b_2, \ldots, b_t)$. Then there exists an orthogonal design of order n and type $(1, b_1, \ldots, b_t)$.*

In order 8 , for example, there can be no amicable orthogonal designs of types $((1, 1, 1); (8))$, since the existence of such a pair would imply the existence of an orthogonal design of order 8 and type $(1, 8)$, which is clearly impossible.

THEOREM 4.2.2 (Wolfe). *Suppose A and B are amicable orthogonal designs of order $n \equiv 0 \pmod 4$, $n \neq 4$, and types $((1, 1, n-2); (b_1, \ldots, b_t))$, then $b_i \neq 1$ for any i .*

We now give more non-existence results.

THEOREM 4.2.3. *There are no amicable orthogonal designs of order $n \equiv 0 \pmod 4$, $n \geq 8$, and types $((1, 1, k); (n))$, k odd.*

Proof. Assume that such an amicable pair, A and B , exists.

By multiplying by suitable monomial matrices if necessary, we may assume

$$A = x_1 I + x_2 X + x_3 Y$$

with

$$X = \bigoplus_{n/2} \begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix} .$$

By considering $XY^t = -YX^t$, $Y^t = -Y$, $XB^t = BX^t$, and $B^t = B$, we see that Y and B are made up of 2×2 blocks of the form

$$\begin{bmatrix} a & b \\ b & \bar{a} \end{bmatrix} , \quad a, b \in \{0, 1, -1\} .$$

Now $(AB)^t = B^t A^t = BA^t = AB^t = AB$, and therefore A and B are amicable only if AB is symmetric, and hence only if the top left hand 2×2 block of AB , and therefore YB , is symmetric.

The top left hand 2×2 block of YB , however, is made up of the sum of 2×2 blocks of the form

$$\begin{bmatrix} x & y \\ y & \bar{x} \end{bmatrix} \begin{bmatrix} a & b \\ b & \bar{a} \end{bmatrix} = \begin{bmatrix} xa+yb & xb-ya \\ -xb+ya & xa+yb \end{bmatrix}$$

with $a, b = \pm 1$ and $x, y \in \{0, 1, -1\}$.

This sum cannot produce a symmetric block unless the sum of the $(ya-xb)$'s is zero, which is clearly impossible with an odd number of non-zero x and y's .

Therefore, there can be no designs, A and B , of the required types.

Before we proceed with the remaining results, we need the following lemma.

LEMMA 4.2.4. *If A and B are amicable orthogonal designs of order $n \equiv 0 \pmod 4$ and types $\big((1, n-1); (1)\big)$ then*

$$B = \bigoplus_{(n/2)-1} Y \oplus X$$

where

$$X = \begin{bmatrix} - & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Proof. From Lemma 4.1.2, we may assume B is symmetric and $A = Ix_1 + Wx_2$, where W is a skew weighing matrix of weight $n - 1$, that is, W is a $(0, 1, -1)$ matrix satisfying $W^t = -W$ and $WW^t = (n-1)I$. It is obvious that we can find a monomial matrix, P , such that

$$PBP^t = \bigoplus_{1 \le i \le n/2} X_i$$

where $X_i = X$ or Y or $\pm I_2$.

If, however, $X_i = \pm I_2$, for any i , then positions $(2i, 2i-1)$ and $(2i-1, 2i)$ of AB cannot be equal. Therefore none of the X_i's are $\pm I_2$. This also means that at most one of the X_i's is $\pm X$, for if two X_i's are $\pm X$, we can find another monomial matrix which produces $\pm I_2$ somewhere on the diagonal of B .

We now assume $X_i = Y$, $1 \le i \le n/2$, and

$$W = \begin{bmatrix} 0 & 1 & & & & \\ - & 0 & A_1 & A_2 & \cdots & A_{(n/2)-1} \\ & \bar{A}_1^t & & & & \\ & \bar{A}_2^t & & & & \\ & \vdots & & & Z & \\ & \bar{A}_{(n/2)-1}^t & & & & \end{bmatrix}$$

where A_i are (2×2) matrices with entries ± 1 .

Since A and B are amicable, we have W and B are amicable, and therefore WB is symmetric.

Let

$$A_1 = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \quad x_i = \pm 1 .$$

Now

$$A_1Y = \begin{bmatrix} x_2 & x_1 \\ x_4 & x_3 \end{bmatrix} \text{ and } A_1^tY = \begin{bmatrix} \bar{x}_3 & \bar{x}_1 \\ \bar{x}_4 & \bar{x}_2 \end{bmatrix},$$

and therefore $x_2 = -x_3$ and $x_1 = -x_4$.

This reasoning is also true for the other A_i's , and so

$$A_i = \pm \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix} \text{ or } \pm \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix} .$$

Now, by the orthogonality of A ,

$$\sum_{i=1}^{(n/2)-1} A_iA_i^t = (n-2)I_2 . \qquad (1)$$

Also

$$A_iA_i^t = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} .$$

But since there is an odd number of A_i's , (1) cannot possibly be true.

Therefore, exactly one of the X_i's is X , and we may assume $X_{n/2} = X$ and $X_i = Y$, $i = 1,\dots,(n/2)-1$.

We now use this lemma to produce a result similar to that of Wolfe (Theorem 4.2.2).

THEOREM 4.2.5. *There are no amicable orthogonal designs of order* $n \equiv 0 \pmod 8$ *and types* $\big((1);\ (1, a, n-a-1)\big)$, $a = 2, 3, 4$ *or* 5 .

Proof. Assume that A and B are amicable orthogonal designs of type $(1, a, n-a-1)$ and (1) respectively.

We may assume B is of the form given in Lemma 4.2.4 and this implies that the first two rows of A have the following form:

$$x_1\ a_2\ a_3\ a_4\ a_5\ a_6\ \dots\ a_{(n/2)-3}\ a_{(n/2)-2}\ a_{(n/2)-1}\ a_{n/2}$$

$$\bar{a}_2\ x_1\ \bar{a}_4\ \bar{a}_3\ \bar{a}_6\ \bar{a}_5\ \dots\ \bar{a}_{(n/2)-2}\ \bar{a}_{(n/2)-3}\ \bar{a}_{(n/2)-1}\ a_{n/2}$$

for some a_i's .

The innerproduct of these two rows gives

$$-2\left(a_3a_4 + \dots + a_{(n/2)-3}a_{(n/2)-2}\right) - a_{(n/2)-1}^2 + a_{n/2}^2 = 0 . \qquad \text{(I)}$$

Let x_2 be the variable which appears a times per row and column and x_3 the remaining variable.

From (I) we have

(i) $a_{(n/2)-1} = \pm a_{n/2}$;

(ii) $a_i = \pm x_2$ for an even number of i's , $3 \le i \le (n/2)-2$;

(iii) $a_{2i-1} = a_{2i} = \pm x_2$ for an even number of i's , $2 \le i \le (n/4)-1$.

We note that similar properties exist for the other rows of A , except the last two.

Now we consider the matrix A_1 obtained from A by putting $x_1 = x_3 = 0$ and $x_2 = 1$ and show that no such matrix can exist if $a = 2, 3, 4$ or 5 .

Case 1. $a = 2$ or 3 .

We can find a monomial, P , such that $PBP^t = B$ and, if $n \ge 16$,

$$PA_1P^t = \left[\begin{array}{c|c|c|c|c}
\begin{matrix} 0 & a_1 \\ \overline{a}_1 & 0 \end{matrix} & 0 & 0 & 0 & \begin{matrix} 1 & 1 \\ - & 1 \end{matrix} \\
\hline
0 & \begin{matrix} 0 & a_2 & & \\ \overline{a}_2 & 0 & & \\ & & 0 & a_2 \\ & & \overline{a}_2 & 0 \end{matrix} & 0 & \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & - & 0 & - \\ 1 & 0 & - & 0 \\ 0 & - & 0 & 1 \end{matrix} & 0 \\
\hline
0 & 0 & A_2 & 0 & 0 \\
\hline
0 & \begin{matrix} - & 0 & - & 0 \\ 0 & 1 & 0 & 1 \\ - & 0 & 1 & 0 \\ 0 & 1 & 0 & - \end{matrix} & 0 & \begin{matrix} 0 & a_2 & & \\ \overline{a}_2 & 0 & & \\ & & 0 & a_2 \\ & & \overline{a}_2 & 0 \end{matrix} & 0 \\
\hline
\begin{matrix} - & 1 \\ - & - \end{matrix} & 0 & 0 & 0 & \begin{matrix} 0 & a_1 \\ \overline{a}_1 & 0 \end{matrix}
\end{array}\right]$$

where

$$a_i = \begin{cases} \pm 1 & \text{if } a = 3 \ , \\ 0 & \text{if } a = 2 \ . \end{cases}$$

By deleting rows and columns 3, 4, 5, 6, n-2, n-3, n-4 and n-5 , we obtain a matrix similar in structure to A_1 but in order $n - 8$. Hence, the existence of A_1 in order n implies the existence of a similar matrix in order 8 . It is easy to see, however, that no such design exists in order 8 .

Therefore, there is no A_1 in order n . This means that there are no amicable orthogonal designs of types $\big((1);\ (1,\ a,\ n\text{-}a\text{-}1)\big)$, $a = 2$ or 3 , in order $n \equiv 0 \pmod 8$.

Case 2. $a = 4$

We can find a monomial, Q , such that $QBQ^t = B$ and, if $n \geq 24$,

$$QA_1Q^t = \left[\begin{array}{c|c|c} 0 & 0 & \begin{matrix} & Z_1 & L \\ Z_1 & Z_2 & \end{matrix} \\ \hline 0 & A_2 & 0 \\ \hline \begin{matrix} & \overline{Z}_1^t \\ \overline{Z}_1^t & \overline{Z}_2^t \\ \overline{L}^t & \end{matrix} & 0 & 0 \end{array}\right]$$

where

$$L = \begin{bmatrix} 1 & 1 \\ - & 1 \\ 1 & 1 \\ - & 1 \end{bmatrix} ,$$

$$Z_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & - & 0 & - \\ - & 0 & - & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and

$$Z_2 = \begin{bmatrix} 1 & 0 & - & 0 \\ 0 & - & 0 & 1 \\ 1 & 0 & - & 0 \\ 0 & - & 0 & 1 \end{bmatrix} .$$

By deleting rows and columns 1, 2, 3, 4, n, n-1, n-2 and n-3 of QA_1Q^t and replacing the first two columns (rows) of the Z_2 $\left(\overline{Z}_2^t\right)$ in the corners by L $\left(\overline{L}^t\right)$ we obtain a matrix with similar structure to A_1 but in order n-8 . Hence, the existence of A_1 in order n implies the existence of a similar matrix in order 16 . It is easy to see that no such designs can exist in order 8 or 16 and so, there can be no amicable designs of types (1) and (1, 4, n-5) in order $n \equiv 0 \pmod 8$.

Case 3. $a = 5$.

We proceed in a similar fashion to that of Case 2 but with diagonal 2×2 blocks of the form $\begin{bmatrix} 0 & a_i \\ \bar{a}_i & 0 \end{bmatrix}$ inserted. We note that this makes no difference to the proof.

COROLLARY 4.2.6. *There are no amicable orthogonal designs of order* $n \equiv 0 \pmod 4$ *and types* $\big((1); (1, a, b, c)\big)$, $a + b + c = n-1$, $a, b, c \neq 0$, *and* abc *odd.*

Proof. By considering (I) in the proof of Theorem 4.2.5 we can see that each variable appears an even number of times off the diagonal 2×2 block and therefore only one of a, b and c is odd.

We now give two results in order 8 .

LEMMA 4.2.7. *There are no amicable orthogonal designs of order* 8 *and types* $\big((1); (2, 2, 2, 2)\big)$.

Proof. By Lemma 4.1.2 we need only prove that there is no symmetric (2,2,2,2) orthogonal design in order 8 . Assume that such a design, B , exists. Each variable must appear an even number of times on the diagonal and we can permute rows and columns of B so that the variable x_1 , say, appears in positions (1, 1), (2, 2), (3, 4) and (4, 3) and one of positions (1, 2), (1, 3) or (1, 4) . It is easy to see, however, that the position (1, 2) is the only one of these which is possible. We may therefore assume $\pm x_1$ appears in position (3, 3) and (4, 4) of B .

Therefore, if a variable appears on the diagonal, it must appear either 4 or 8 times. Hence we may assume B is

$$\begin{bmatrix} \begin{matrix} x_1 & x_1 & x_2 & \bar{x}_2 \\ x_1 & \bar{x}_1 & \bar{x}_2 & \bar{x}_2 \\ x_2 & \bar{x}_2 & x_1 & x_1 \\ \bar{x}_2 & \bar{x}_2 & x_1 & \bar{x}_1 \end{matrix} & X \\ X^t & \begin{matrix} x_1 & x_1 & x_2 & \bar{x}_2 \\ x_1 & \bar{x}_1 & \bar{x}_2 & \bar{x}_2 \\ x_2 & \bar{x}_2 & x_1 & x_1 \\ \bar{x}_2 & \bar{x}_2 & x_1 & \bar{x}_1 \end{matrix} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \begin{matrix} x_1 & x_1 & x_2 & \bar{x}_2 \\ x_1 & \bar{x}_1 & \bar{x}_2 & \bar{x}_2 \\ x_2 & \bar{x}_2 & x_1 & x_1 \\ \bar{x}_2 & \bar{x}_2 & x_1 & \bar{x}_1 \end{matrix} & Y \\ Y^t & \begin{matrix} x_2 & x_2 & x_1 & \bar{x}_1 \\ x_2 & \bar{x}_2 & \bar{x}_1 & \bar{x}_1 \\ x_1 & \bar{x}_1 & x_2 & x_2 \\ \bar{x}_1 & \bar{x}_1 & x_2 & \bar{x}_2 \end{matrix} \end{bmatrix}$$

for some X and Y .

But no such X or Y , with 2 variables appearing twice per row and column, can exist if B is to be orthogonal.

Therefore, there is no symmetric (2, 2, 2, 2) orthogonal design of order 8 .

LEMMA 4.2.8. *There are no amicable orthogonal designs of order* 8 *and types* $\big((1, 2, 2, 3); (4, 4)\big)$.

Proof. We assume that there are amicable orthogonal designs A and B of types (1, 2, 2, 3) and (4, 4) respectively.

As before, we may assume $A = Ix + X$, $X^t = -X$, $B^t = B$ and $XB^t = BX^t$. In fact, XB is symmetric.

It is obvious that we can find permutations and negations of rows and columns of X and B which allow us to assume X has the following form:

$$\begin{bmatrix} 0 & a & x_1 & x_1 & x_2 & b & c & d \\ \bar{a} & 0 & x_1 & \bar{x}_1 & b & \bar{x}_2 & d & \bar{c} \\ \bar{x}_1 & \bar{x}_1 & 0 & a & d & c & \bar{b} & \bar{x}_2 \\ \bar{x}_1 & x_1 & \bar{a} & 0 & c & \bar{d} & \bar{x}_2 & b \\ \bar{x}_2 & \bar{b} & \bar{d} & \bar{c} & 0 & a & x_1 & x_1 \\ \bar{b} & x_2 & \bar{c} & d & \bar{a} & 0 & x_1 & \bar{x}_1 \\ \bar{c} & \bar{d} & b & x_2 & \bar{x}_1 & \bar{x}_1 & 0 & a \\ \bar{d} & c & x_2 & \bar{b} & \bar{x}_1 & x_1 & \bar{a} & 0 \end{bmatrix} \qquad \text{(I)}$$

for some a, b, c, d .

Now, the variable x_2 appears again in the first row. That is $\pm x_2 \in \{a, b, c, d\}$.

If $d = \pm x_2$, we interchange rows and columns 3 and 4 and rows and columns 7 and 8 . We then negate rows and columns 2 and 6 . These operations shift $\pm x_2$ from d to c without essentially changing the rest of the matrix.

If $a = \pm x_2$, we use the permutation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & - \end{bmatrix}$$

and interchange the names of the variables x_1 and x_2 . This gives the matrix (I) with $c = \pm x_2$.

If $c = -x_2$, we negate rows and columns 3, 4, 7 and 8 and change $-x_1$ to x_1 . This gives (I) with $c = x_2$.

If $b = -x_2$, we interchange rows and columns 1 and 2 , and, 5 and 6 . Then we negate rows and columns 4 and 8 , and change $-x_2$ to x_2 . We obtain (I) with $b = x_2$.

Therefore X can be written in the form (I) with $b = x_2$ or $c = x_2$.

We let $X[t_1, t_2, t_3]$ be X with x_i replaced by t_i and consider the case: $b = x_2$. We have

$$X[x_1,\ x_2,\ 0] = \begin{bmatrix} 0 & 0 & x_1 & x_1 & x_2 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & \bar{x}_1 & x_2 & \bar{x}_2 & 0 & 0 \\ \bar{x}_1 & \bar{x}_1 & 0 & 0 & 0 & 0 & \bar{x}_2 & \bar{x}_2 \\ \bar{x}_1 & x_1 & 0 & 0 & 0 & 0 & \bar{x}_2 & x_2 \\ \bar{x}_2 & \bar{x}_2 & 0 & 0 & 0 & 0 & x_1 & x_1 \\ \bar{x}_2 & x_2 & 0 & 0 & 0 & 0 & x_1 & \bar{x}_1 \\ 0 & 0 & x_2 & x_2 & \bar{x}_1 & \bar{x}_1 & 0 & 0 \\ 0 & 0 & x_2 & \bar{x}_2 & \bar{x}_1 & x_1 & 0 & 0 \end{bmatrix} .$$

Now let

$$B = \left[\begin{array}{c|c} B_1 & \begin{array}{c|c} B_3 & \begin{array}{cc} a & b \\ c & d \end{array} \\ \hline \begin{array}{cc} e & f \\ g & h \end{array} & B_4 \end{array} \\ \hline \begin{array}{c|c} B_3^t & \begin{array}{cc} e & g \\ f & h \end{array} \\ \hline \begin{array}{cc} a & c \\ b & d \end{array} & B_4^t \end{array} & B_2 \end{array}\right]$$

for some matrices B_i , $i = 1, \ldots, 4$.

In order for XB to be symmetric, $X[1, 0, 0]B$ and $X[0, 1, 0]B$ must be symmetric.

The $(1, 5)$ and $(5, 1)$ entries of $X[1, 0, 0]B$ give

$$e + g = a + b$$

while the $(5, 7)$ and $(7, 5)$ entries of $X[0, 1, 0]B$ give

$$e + g = -a - c .$$

These equations imply

$$a = -b = -c \quad \text{and} \quad e = -g .$$

By considering the $(4, 7)$, $(7, 4)$, $(3, 7)$ and $(7, 3)$ entries of $X[1, 0, 0]B$, we obtain

$$a = -b = -c = g = -e = f = h .$$

By using these results, we can see that the $(1, 4)$ and $(4, 1)$ entries of $X[0, 1, 0]B$ cannot possibly be equal. Therefore $b \neq x_2$.

If $c = x_2$ we have

$$X = \begin{bmatrix} 0 & x_3 & x_1 & x_1 & x_2 & x_3 & x_2 & ex_3 \\ \overline{x}_3 & 0 & x_1 & \overline{x}_1 & x_3 & \overline{x}_2 & ex_3 & \overline{x}_2 \\ \overline{x}_1 & \overline{x}_1 & 0 & x_3 & ex_3 & x_2 & \overline{x}_3 & \overline{x}_2 \\ \overline{x}_1 & x_1 & \overline{x}_3 & 0 & x_2 & \overline{ex}_3 & \overline{x}_2 & x_3 \\ \overline{x}_2 & \overline{x}_3 & \overline{ex}_3 & \overline{x}_2 & 0 & x_3 & x_1 & x_1 \\ \overline{x}_3 & x_2 & \overline{x}_2 & ex_3 & \overline{x}_3 & 0 & x_1 & \overline{x}_1 \\ \overline{x}_2 & \overline{ex}_3 & x_3 & x_2 & \overline{x}_1 & \overline{x}_1 & 0 & x_3 \\ \overline{ex}_3 & x_2 & x_2 & \overline{x}_3 & \overline{x}_1 & x_1 & \overline{x}_3 & 0 \end{bmatrix},$$

with $e = \pm 1$.

We put

$$B = \begin{bmatrix} B_1 & B_2 \\ B_2^t & B_3 \end{bmatrix},$$

for some B_i, $i = 1, 2$ and 3, and consider $X[1, 0, 0]B$ and $X[0, 1, 0]B$. This gives

$$B_1 = \begin{bmatrix} a & \overline{a} & \overline{b} & \overline{b} \\ \overline{a} & \overline{a} & \overline{b} & b \\ \overline{b} & \overline{b} & a & \overline{a} \\ \overline{b} & b & \overline{a} & \overline{a} \end{bmatrix}$$

and

$$B_3 = \begin{bmatrix} b & b & \overline{a} & \overline{a} \\ b & \overline{b} & \overline{a} & a \\ \overline{a} & \overline{a} & \overline{b} & \overline{b} \\ \overline{a} & a & \overline{b} & b \end{bmatrix}$$

for some a and b.

By considering XB, we obtain

$$B_2 = \begin{bmatrix} m & \overline{n} & p & q \\ n & m & \overline{q} & p \\ p & \overline{q} & \overline{m} & \overline{n} \\ q & p & n & \overline{m} \end{bmatrix}$$

for some m, n, p and q .

Entries (1, 3) and (3, 1) of $X[0, 1, 0]B$ give

$$m - p = n + q$$

and entries (1, 5), (5, 1), (1, 6) and (6, 1) of $X[0, 0, 1]$ give

$$2n + b(1-e) - a(1+e) = 0$$

$$2m - b(1-e) + a(1+e) = 0 .$$

These give

$$m = -n = p = q = b \quad \text{if} \quad e = -1 ,$$

$$m = -n = p = q = -a \quad \text{if} \quad e = 1 .$$

Therefore, one variable appears at least six times per row and column, and hence there is no B of the required type.

This completes the proof.

We use the above results to obtain:

LEMMA 4.2.9. *There are no amicable orthogonal designs of order* 8 *and types*

$\big((1, 1, 1, 5); (8)\big)$ $\big((1, 7); (1, 1, 6)\big)$

$\big((1, 7); (5)\big)$ $\big((1, a, b); (1, 7)\big)$, $a + b = 7$, a *and* $b \neq 0$

$\big((1, 1, 3, 3); (8)\big)$ $\big((1, 7); (2, 2, 2, 2)\big)$

$\big((1, 2, 2, 3); (4, 4)\big)$.

Proof. Wolfe (Theorems 4.2.1 and 4.2.2) shows that there are no amicable designs of type $\big((1, 1, 1, 5); (8)\big)$ and $\big((1, 7); (1, 1, 6)\big)$ and, in [17], which is bound into the back of this thesis, we prove that there are no amicable designs of type $\big((1, 7); (5)\big)$. The remainder of the proof is obtained by appealing to Theorem 4.2.5, Theorem 4.2.3, Lemmas 4.2.7 and 4.2.8.

LEMMA 4.2.10 (Robinson and Wallis). *There are no amicable orthogonal designs of order* 16 *and types* $((1, 15); (1))$.

Proof. As before, we may assume B is symmetric and $A = Ix_1 + Wx_2$, where W is a skew weighing matrix of weight 15 .

We now consider Lemma 4.2.4 and rearrange rows and columns so that

$$B = \begin{bmatrix} X & & & 0 \\ & Y & & \\ & & Y & \\ 0 & & & \ddots \end{bmatrix},$$

where $X = \begin{bmatrix} - & 0 \\ 0 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We also put

$$W = \begin{bmatrix} \begin{matrix} 0 & 1 \\ - & 0 \end{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \\ -A_1^t & \begin{matrix} 0 & a \\ -a & 0 \end{matrix} & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \\ \vdots & \cdots & . & . & . & . & . & . \end{bmatrix}.$$

Since WB must be symmetric, we see each A_i is of the form $\begin{bmatrix} c & c \\ d & -d \end{bmatrix}$ and each B_j is of the form $\begin{bmatrix} e & f \\ -f & -e \end{bmatrix}$. Clearly we can choose $A_i = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} = V$ and $B_j = Z, -Z, Z^t$ or $-Z^t$ where $Z = \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix}$. Now $VZ^t = \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}$ and $VZ = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$. So the orthogonality condition of the first and second rows of blocks gives us $a = -1$ and

$$\{B_2, B_3, B_4, B_5, B_6, B_7\} = \{Z, -Z^t, Z, -Z, Z^t, -Z^t\} .$$

Indeed, each diagonal block of A , except the first, can be written as

$D = \begin{bmatrix} 0 & - \\ 1 & 0 \end{bmatrix}$. Clearly since two Z's occur in each row we can simultaneously apply monomial matrices P to W and B so that PWP^t is the W given below and B is unchanged:

$$W = \begin{bmatrix} D^t & V & V & V & V & V & V & V \\ -V^t & D & Z & Z & & & & \\ -V^t & -Z^t & D & & & & & \\ -V^t & -Z^t & & D & & & & \\ \vdots & . & . & . & . & . & . & . \end{bmatrix}.$$

We now observe that two rows of blocks can have one of the following:

(i) $\begin{matrix} -V^t & D & Z \\ -V^t & -Z^t & D \end{matrix}$ which have inner product $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

(ii) $\begin{matrix} -V^t & D & -Z \\ -V^t & Z^t & D \end{matrix}$ which have inner product $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$;

(iii) $\begin{matrix} -V^t & D & Z^t \\ -V^t & -Z & D \end{matrix}$ which have inner product $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$;

(iv) $\begin{matrix} -V^t & D & -Z^t \\ -V^t & Z & D \end{matrix}$ which have inner product $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Now the inner products of $\begin{matrix} Z \\ Z \end{matrix}$, $\begin{matrix} Z \\ Z^t \end{matrix}$, $\begin{matrix} Z^t \\ Z^t \end{matrix}$, $\begin{matrix} Z^t \\ Z \end{matrix}$, are $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ respectively.

We now choose each of the choices (i), (ii), (iii), (iv) for the third and fourth rows of blocks and try to complete the matrix given that there must be Z, Z, $-Z$, Z^t, $-Z^t$, $-Z^t$ in each row.

Case (i). Row 3 and 4 can only be completed (ignoring column permutations) as shown -

$$\begin{array}{cccccccc} -V^t & D & Z & Z & & & & \\ -V^t & -Z^t & D & Z & Z & -Z & -Z^t & Z^t \\ -V^t & -Z^t & -Z^t & D & Z & Z & Z^t & -Z \end{array} .$$

The only choice to complete the second row of blocks to give orthogonality with the third is $-Z\ Z^t\ -Z^t\ -Z^t$ and now the second and fourth rows of blocks are not orthogonal.

Case (ii). Rows 3 and 4 can be partially completed in only one way (ignoring column permutations) to enforce orthogonality as shown -

$$\begin{array}{cccccccc} -V^t & -Z^t & D & -Z & Z & Z^t & -Z^t \\ -V^t & -Z^t & Z^t & D & -Z & -Z^t & Z^t \end{array} .$$

But now the fourth row of blocks has too many $\pm Z^t$'s .

Cases (iii) and (iv) can be converted into cases (ii) and (i) respectively by permuting the third and fourth rows of blocks and third and fourth columns of blocks of both W and B simultaneously.

Hence the matrix W cannot be completed and we have the lemma.

4.3 Constructing Amicable Orthogonal Designs

A very important construction is given by Wolfe [27] and we now state this result.

THEOREM 4.3.1 (Wolfe). *If there exist amicable orthogonal designs of order n and types $((a_1, \ldots, a_s); (b_1, \ldots, b_t))$ and a pair of amicable orthogonal designs of order m and types $((c_1, \ldots, c_u); (d_1, \ldots, d_v))$, then there exists a pair of amicable orthogonal designs of order nm and types*
$$((b_1c_1, \ldots, b_1c_{u-1}, a_1c_u, \ldots, a_sc_u); (b_1d_1, \ldots, b_1d_v, b_2c_u, \ldots, b_tc_u)) .$$

The following corollaries are noted by Geramita and Wallis [10].

COROLLARY 4.3.2. *If there are amicable orthogonal designs of order* n *and types* $\big((a_1, \ldots, a_s); (b_1, \ldots, b_t)\big)$ *and a pair of amicable designs of types* $\big((1, w); (m)\big)$ *in order* p , *then there is a pair of amicable orthogonal designs of order* pn *and types*

$$\big((a_1, wa_1, ma_2, \ldots, ma_s); (mb_1, mb_2, \ldots, mb_t)\big) .$$

COROLLARY 4.3.3. *Suppose that there exists amicable orthogonal designs of order* n *and types* $\big((a_1, \ldots, a_s); (b_1, \ldots, b_t)\big)$. *Then there exist amicable orthogonal designs of order* $2n$ *and types*

$$\big((a_1, a_1, 2a_2, \ldots, 2a_s); (2b_1, 2b_2, \ldots, 2b_t)\big) .$$

Proof. We simply note that

$$\begin{bmatrix} x_1 & x_2 \\ \bar{x}_2 & x_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_1 & y_2 \\ y_2 & \bar{y}_1 \end{bmatrix}$$

are amicable orthogonal designs of order 2 and types $\big((1, 1); (1, 1)\big)$. We equate two variables in one design and use the resulting designs in the above corollary.

Before we proceed we note that

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_3 \\ \bar{x}_2 & x_1 & x_3 & \bar{x}_3 \\ \bar{x}_3 & \bar{x}_3 & x_1 & x_2 \\ \bar{x}_3 & x_3 & \bar{x}_2 & x_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_1 & y_2 & y_3 & y_3 \\ y_2 & \bar{y}_1 & y_3 & \bar{y}_3 \\ y_3 & y_3 & \bar{y}_2 & \bar{y}_1 \\ y_3 & \bar{y}_3 & \bar{y}_1 & y_2 \end{bmatrix}$$

are amicable designs of order 4 and types $\big((1, 1, 2); (1, 1, 2)\big)$.

Using these designs in Corollary 4.3.3 and proceeding by induction we obtain:

COROLLARY 4.3.4. *There exist amicable orthogonal designs of order* 2^t *and types* $\big((1, 1, 2, 4, \ldots, 2^{t-1}); (2^{t-2}, 2^{t-2}, 2^{t-1})\big)$.

LEMMA 4.3.5. *The following are the types of amicable orthogonal designs in order* 8 :

(i) $\big((1, 1, 2, 2, 2); (8)\big)$;

(ii) $\big((1, 2, 2, 3); (2, 6)\big)$;

(iii) $\big((1, 7); (1, 7)\big)$;

(iv) $((1,1,2,4);(2,2,4))$.

The above designs are given in [17] which is bound in the rear of this thesis.

CHAPTER 5

PRODUCT DESIGNS

This chapter is concerned with generalizing a construction of Geramita and Wallis. We define and construct product designs and show how these designs may be combined with amicable orthogonal designs to produce new orthogonal designs.

5.1 Introduction

In 4.1 we noted that amicable orthogonal designs were introduced as a way of constructing new orthogonal designs. In order to exploit the structure of these designs more fully, Geramita and Wallis, in [10], produced the following construction:

CONSTRUCTION 5.1.1 (Geramita and Wallis). *Suppose there exist three matrices* R, P *and* S *of order* n *which give amicable orthogonal designs*

$$S\ ,\quad x_1R + x_2P$$

of types $\left((s_1, \dots, s_t); (u_1, u_2)\right)$. *Then*

$$\begin{bmatrix} y_1R+y_2P & y_3R+y_4P & S & y_5R+y_6P \\ -y_3R+y_4P & y_1R-y_2P & -y_5R-y_6P & S \\ -S & y_5R-y_6P & y_1R+y_2P & -y_3R+y_4P \\ -y_5R+y_6P & -S & y_3R+y_4P & y_1R-y_2P \end{bmatrix}$$

is an orthogonal design of order $4n$ *and type*

$$\left(s_1, s_2, \dots, s_t, u_1, u_1, u_1, u_2, u_2, u_2\right) .$$

We note that the above matrix can be written in the form

$$M_1 \times R + M_2 \times P + N \times S$$

where M_1, M_2 and N are the 4×4 matrices of the coefficients of R, P and S respectively.

In this chapter we are interested in generalizing this idea. With this in mind, we give the following definition.

DEFINITION 5.1.2. Let M_1, M_2 and N be orthogonal designs of order n and types $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_s)$ and $(c_1, \ldots, c_t)$ respectively. Then $(M_1; M_2; N)$ are *product designs* of order n and types $(a_1, \ldots, a_r; b_1, \ldots, b_s; c_1, \ldots, c_t)$ if

(i) $M_1 * N = M_2 * N = 0$ (Hadamard product),

(ii) $M_1 + N$ and $M_2 + N$ are orthogonal designs, and

(iii) $M_1 M_2^t = M_2 M_1^t$.

Construction 5.1.1 produces product designs of order 4 and types (1, 1, 1; 1, 1, 1; 1) .

5.2. Constructing product designs

Before we give the main results of this section, we produce two examples of product designs. For convenience we write the designs M_1, M_2 and N in the form $M_1 + zN$ and M_2 .

EXAMPLE 5.2.1. The following matrices give product designs of order 8 and types (1, 1, 1; 1, 1, 1; 5) .

$$\begin{bmatrix} \bar{x}_1 & x_2 & z & x_3 & z & z & z & \bar{z} \\ \bar{x}_2 & x_1 & \bar{x}_3 & z & z & \bar{z} & z & z \\ \bar{z} & x_3 & x_1 & \bar{x}_2 & z & \bar{z} & \bar{z} & \bar{z} \\ \bar{x}_3 & \bar{z} & x_2 & x_1 & z & z & \bar{z} & z \\ \bar{z} & \bar{z} & \bar{z} & \bar{z} & x_1 & x_2 & z & \bar{x}_3 \\ \bar{z} & z & z & \bar{z} & \bar{x}_2 & x_1 & x_3 & z \\ \bar{z} & \bar{z} & z & z & \bar{z} & \bar{x}_3 & x_1 & \bar{x}_2 \\ z & \bar{z} & z & \bar{z} & x_3 & \bar{z} & x_2 & x_1 \end{bmatrix} \quad \begin{bmatrix} \bar{y}_1 & y_2 & 0 & y_3 & & & & \\ y_2 & \bar{y}_1 & \bar{y}_3 & 0 & & & 0 & \\ 0 & \bar{y}_3 & y_1 & y_2 & & & & \\ y_3 & 0 & y_2 & \bar{y}_1 & & & & \\ & & & & y_2 & y_3 & 0 & \bar{y}_1 \\ & & 0 & & y_3 & \bar{y}_2 & y_1 & 0 \\ & & & & 0 & y_1 & y_2 & y_3 \\ & & & & \bar{y}_1 & 0 & y_3 & \bar{y}_2 \end{bmatrix} .$$

EXAMPLE 5.2.2. The following matrices give product designs of order 12 and types $(1, 1, 1; 1, 1, 1; 9)$.

$$
\begin{bmatrix}
x_1 & x_2 & x_3 & \bar{z} & z & z & z & \bar{z} & z & \bar{z} & z & \bar{z} \\
\bar{x}_2 & x_1 & \bar{z} & \bar{x}_3 & z & \bar{z} & \bar{z} & \bar{z} & \bar{z} & \bar{z} & \bar{z} & \bar{z} \\
\bar{x}_3 & z & x_1 & x_2 & z & z & \bar{z} & z & z & z & \bar{z} & \bar{z} \\
z & x_3 & \bar{x}_2 & x_1 & z & \bar{z} & z & z & z & \bar{z} & \bar{z} & z \\
\bar{z} & \bar{z} & \bar{z} & \bar{z} & x_1 & x_2 & x_3 & \bar{z} & z & z & \bar{z} & z \\
\bar{z} & z & \bar{z} & z & \bar{x}_2 & x_1 & \bar{z} & \bar{x}_3 & z & \bar{z} & z & z \\
\bar{z} & z & z & \bar{z} & \bar{x}_3 & z & x_1 & x_2 & \bar{z} & \bar{z} & \bar{z} & z \\
z & z & \bar{z} & \bar{z} & z & x_3 & \bar{x}_2 & x_1 & \bar{z} & z & z & z \\
\bar{z} & z & \bar{z} & \bar{z} & \bar{z} & \bar{z} & z & z & x_1 & x_2 & x_3 & \bar{z} \\
z & z & \bar{z} & z & \bar{z} & z & z & \bar{z} & \bar{x}_2 & x_1 & \bar{z} & \bar{x}_3 \\
\bar{z} & z & z & z & z & \bar{z} & z & \bar{z} & \bar{x}_3 & z & x_1 & x_2 \\
z & z & z & \bar{z} & \bar{z} & \bar{z} & \bar{z} & \bar{z} & z & x_3 & \bar{x}_2 & x_1
\end{bmatrix}
\begin{bmatrix}
\begin{matrix} y_1 & y_2 & y_3 & 0 \\ y_2 & \bar{y}_1 & 0 & \bar{y}_3 \\ y_3 & 0 & \bar{y}_1 & y_2 \\ 0 & \bar{y}_3 & y_2 & y_1 \end{matrix} & 0 & 0 \\
0 & \begin{matrix} y_2 & y_3 & y_1 & 0 \\ y_3 & \bar{y}_2 & 0 & \bar{y}_1 \\ y_1 & 0 & \bar{y}_2 & y_3 \\ 0 & \bar{y}_1 & y_3 & y_2 \end{matrix} & 0 \\
0 & 0 & \begin{matrix} y_3 & \bar{y}_1 & \bar{y}_2 & 0 \\ \bar{y}_1 & \bar{y}_3 & 0 & y_2 \\ \bar{y}_2 & 0 & \bar{y}_3 & \bar{y}_1 \\ 0 & y_2 & \bar{y}_1 & y_3 \end{matrix}
\end{bmatrix} .
$$

The next theorem gives us a way of obtaining product designs from product designs of smaller orders.

THEOREM 5.2.3. *Let* $(M_1;\ y_1M_3+y_2M_4;\ N)$ *be product designs of order* n *and types* $(a_1, \ldots, a_r;\ b_1, b_2;\ c_1, \ldots, c_t)$ *and let* S *and* $x_1R + x_2P$ *be amicable orthogonal designs of order* m *and types* $((u);\ (v, w))$. *Then*

$$(x_1P \times M_3 + R \times M_1;\ y_1S \times M_3 + y_2R \times M_4;\ z_1P \times M_4 + S \times N)$$

are product designs of order mn *and types*

$$(wb_1, va_1, \ldots, va_r;\ ub_1, vb_2;\ wb_2, uc_1, \ldots, uc_t) .$$

Proof. By straightforward verification.

A very useful form of this theorem is obtained by using the amicable orthogonal designs of order 2 and types $((1, 1); (2))$ (see Geramita, Geramita and Wallis [7]). We state this particular case in the following corollary.

COROLLARY 5.2.4. *Let* $(M_1;\ y_1M_3+y_2M_4;\ N)$ *be product designs of order*

n and types $(a_1, \ldots, a_r; b_1, b_2; c_1, \ldots, c_t)$ *. Then there are product designs of order* $2n$ *and types*

$$(b_1, a_1, \ldots, a_r; 2b_1, b_2; b_2, 2c_1, \ldots, 2c_t) .$$

In Theorem 5.2.3 and Corollary 5.2.4 we may have M_3 or M_4 equal to zero. In this case, however, the next theorem gives a better result.

THEOREM 5.2.5. *If* $(M_1; M_2; N)$ *are product designs of order* n *and types* $(a_1, \ldots, a_r; b; c_1, \ldots, c_t)$ *and if* S *and* $y_1R + P$ *are amicable orthogonal designs of order* m *and types* $((u_1, \ldots, u_l); (v, w_1, \ldots, w_k))$ *then there are product designs of order* mn *and the following types:*

(i) $(va_1, \ldots, va_r; vb; cu_1, \ldots, cu_l, bw_1, \ldots, bw_k)$ *and*

(ii) $(va_1, \ldots, va_r; vb; uc_1, \ldots, uc_t, bw_1, \ldots, bw_k)$,

where u *and* c *are the sums of the* u_i*'s and* c_i*'s respectively.*

Proof. We consider

$$(R \times M_1; R \times M_2; S \times N + P \times M_2)$$

with the appropriate variables equated.

We now produce a method for constructing product designs from amicable orthogonal designs.

THEOREM 5.2.6. *If* S_1 *and* $x_1R_1 + x_2P_1$ *are amicable orthogonal designs of order* n *and types* $((u_1, \ldots, u_l); (v, w))$ *and* S_2 *and* $y_1R_2 + y_2P_2$ *are amicable orthogonal designs of order* m *and types* $((s_1, \ldots, s_k); (r, p))$ *then*

$$(S_1 \times R_2 + xR_1 \times P_2; R_1 \times S_2 + yP_1 \times R_2; P_1 \times P_2)$$

are product designs of order mn *and types*

$$(ru_1, \ldots, ru_l, vp; vs_1, \ldots, vs_k, wr; wp) .$$

Proof. By straightforward verification.

In the following lemma, we give an example of product designs which will be used in the next section to produce a very useful orthogonal design.

LEMMA 5.2.7. *There are product designs of order* 2^t , $t \geq 4$, *and types*

$$\left(1, 1, 1, 1, 2, 4, \ldots, 2^{t-3}; 2, 2^{t-2}; 2, 4, \ldots, 2^{t-3}, 2^{t-2}, 2^{t-2}\right) .$$

Proof. The product designs of order 4 and types (1, 1, 1; 1, 2; 1) produce product designs of order 8 and types (1, 1, 1, 1; 2, 2; 2, 2) (Corollary 5.2.4). This design in turn produces product designs of order 16 and types (1, 1, 1, 1, 2; 2, 4; 2, 4, 4) .

By repeated use of Corollary 5.2.4 we obtain the required result.

In Appendix I we give a list of product designs of orders 4, 8, 16, 32 and 64 which are obtained by using the results given in this section.

5.3 Constructing orthogonal designs from product designs

We now produce a generalization of Construction 5.1.1.

THEOREM 5.3.1. *Let* $y_1R + P$ *and* S *be amicable orthogonal designs of order* m *and types* $\left((v, w_1, \ldots, w_k); (u_1, \ldots, u_l)\right)$ *and let* $(M_1; M_2; N)$ *be product designs of order* n *and types* $(a_1, \ldots, a_r; b_1, \ldots, b_s; c_1, \ldots, c_t)$. *Then there exist orthogonal designs of order* mn *and types*

(i) $(va_1, \ldots, va_r, wb_1, \ldots, wb_s, uc_1, \ldots, uc_t)$,

(ii) $(va_1, \ldots, va_r, wb_1, \ldots, wb_s, u_1c, \ldots, u_lc)$,

(iii) $(va_1, \ldots, va_r, w_1b, \ldots, w_kb, uc_1, \ldots, uc_t)$,

(iv) $(va_1, \ldots, va_r, w_1b, \ldots, w_kb, u_1c, \ldots, u_lc)$,

where b, c, u *and* w *are the sums of the* b_i*'s*, c_i*'s*, u_i*'s and* w_i*'s respectively.*

Proof. We consider

$$M_1 \times R + M_2 \times P + N \times S$$

with the appropriate variables equated.

As an example of the use of this theorem, we give the following lemma.

LEMMA 5.3.2. *There is an orthogonal design of order* 2^t , $t \geq 2$, *and type* $(1, 1, 1, 1, 2, 2, 4, 4, \ldots, 2^{t-2}, 2^{t-2})$.

Proof. If $t \geq 5$ we apply the above theorem with the product designs of order 2^{t-1} and types

$$\left(1, 1, 1, 1, 2, 4, \ldots, 2^{t-4}; 2, 2^{t-3}; 2, 4, \ldots, 2^{t-4}, 2^{t-3}, 2^{t-3}\right)$$

(Lemma 5.2.7) and amicable orthogonal designs of order 2 and types $\left((1, 1); (2)\right)$.

The orthogonal design of order 16 and type (1, 1, 1, 1, 2, 2, 4, 4) may be obtained in a similar manner by using the product designs of order 8 and types (1, 1, 1, 1; 2, 2; 2, 2) (see proof of Lemma 5.2.7).

The design of order 4 and type (1, 1, 1, 1) is given in Geramita *et al* [7] and the design of type (1, 1, 1, 1, 2, 2) is obtained from the orthogonal design of order 8 and type (1, 1, 1, 1, 1, 1, 1, 1) (see Geramita *et al* [7]). This completes the proof.

We note that the above orthogonal design has $2t$ variables. If $t = 4k + 1$, $\rho(t) = 8k + 2 = 2t$ and if $t = 4k + 2$, $\rho(t) = 8k + 4 = 2t$. Therefore, if $t = 4k + 1$ or $4k + 2$, the above design has the maximum number of variables allowed. We also note that the above design is full. That is, the design contains no zeros.

By equating variables in the above design we obtain:

COROLLARY 5.3.3. *All orthogonal designs of type* $(1, 1, a, b, c)$, $a + b + c = 2^t - 2$, *exist in order* 2^t , $t \geq 3$.

In Appendix II we give the types of some orthogonal designs of orders 32, 64 and 128 obtained from product designs. These designs are used in the following section.

We now give two results to show how product designs may be used to obtain orthogonal designs in orders other than powers of 2 .

LEMMA 5.3.4. *There are product designs of order* 12 *and types* (1, 1, 1; 1, 1, 4; 4), (1, 1, 4; 1, 1, 1; 1), (1, 1, 4; 1, 4, 4; 1), (1, 1, 4; 1, 1, 4; 4), (1, 4, 4; 1, 4, 4; 1) *and* (1, 1, 1; 1, 1, 1; 9) .

Proof. In his PhD dissertation, Wolfe gives amicable orthogonal designs of order 6 and types $\big((1, 1); (1, 4)\big)$ and $\big((1, 4); (1, 4)\big)$. By using these designs in Theorem 5.2.6, we obtain the first 5 designs. The last is given in Example 5.2.2.

By using Theorem 5.3.1 with the above designs, we obtain:

COROLLARY 5.3.5. *There are orthogonal designs of order* 24 *and types*

(1, 1, 1, 1, 1, 1, 9, 9)	(1, 1, 1, 1, 4, 4, 4, 4)
(1, 1, 1, 1, 1, 4, 4, 4)	(1, 1, 1, 1, 1, 1, 4, 4) .

These designs are discussed in [18].

5.4 Orthogonal designs in powers of two

Wallis, in [24], gives the following result.

LEMMA 5.4.1. *If there is an orthogonal design of order* n *and type* $(u_1, \ldots, u_s)$ *, then there is an orthogonal design of order* $2n$ *and type* $(u_1, u_1, 2u_2, \ldots, 2u_s)$.

By equating variables in the orthogonal designs given in Appendix II and by using the above lemma, we obtain the following results.

LEMMA 5.4.2. *All* 6*-tuples of the form*

$$(a, b, c, d, e, 32\text{-}a\text{-}b\text{-}c\text{-}d\text{-}e)\ , \quad 0 < a + b + c + d + e < 32\ ,$$

are the types of orthogonal designs of order 32 *, except possibly* (1, 1, 1, 1, 1, 27) .

COROLLARY 5.4.3. *All possible* 5*-tuples, except possibly*

(1, 1, 1, 1, 27) , *are the types of orthogonal designs of order* 32 .

LEMMA 5.4.3. *All* 5*-tuples of the form*

$$(a,\ b,\ c,\ d,\ 2^t-a-b-c-d)\ ,\quad 0 < a + b + c + d < 2^t\ ,$$

are the types of orthogonal designs of order 2^t , $t = 6$ *and* 7 .

COROLLARY 5.4.4. *All possible* n*-tuples,* $n = 1, 2, 3, 4$, *are the types of orthogonal designs of orders* 4, 8, 16, 32, 64 *and* 128 .

Proof of theorem 5.2.3.

Write the new design as $(M_1\ ; M_2\ ; N\ )$.

We want to show :

(i) $M_1 * N = M_2 * N = 0$;

(ii) $M_1 + N$ and $M_2 + N$ are orthogonal designs ; and

(iii) $M_1 M_2^t = M_2 M_1^t$.

For (i) we have

$$M_1 * N = (x_1 P\times M_3 + R\times M_1) * (z_1 P\times M_4 + S\times N)$$
$$= x_1 (P*P)\times(M_3 * M_4) + x_1 (P*S)\times(M_3 * N) + z_1 (R*P)\times(M_1 * M_4) + (R*S)\times(M_1 * N).$$

Since $M_3 * M_4 = M_3 * N = R*P = M_1 * N = 0$, we have $M_1 * N = 0$.

By similar reasoning , we obtain $M_2 * N = 0$.

For the first part of (ii) it is sufficient to show M_1 and N are orthogonal designs and $M_1 N^t = -NM_1^t$.

Now ,
$$M_1 M_1^t = (x_1 P\times M_3 + R\times M_1)(x_1 P^t\times M_3^t + R^t\times M_1^t)$$
$$= x_1^2 PP^t\times M_3 M_3^t + RR^t\times M_1 M_1^t + x_1 (PR^t\times M_3 M_1^t + RP^t\times M_1 M_3^t).$$

Therefore M_1 is an orthogonal design since P , R , M_3 , and M_1 are orthogonal designs , $PR^t = -RP^t$, and , $M_1 M_3^t = M_3 M_1^t$.

Similar reasoning shows N is an orthogonal design.

We also have

$$M_1 N^t = (x_1 P\times M_3 + R\times M_1)(z_1 P^t\times M_4^t + S^t\times N^t)$$
$$= x_1 z_1 PP^t\times M_3 M_4^t + x_1 PS^t\times M_3 N^t + z_1 RP^t\times M_1 M_4^t + RS^t\times M_1 N^t$$
$$= x_1 z_1 PP^t\times(-M_3 M_4^t) + x_1 SP^t\times(-NM_3^t) + z_1(-PR^t)\times M_4 M_1^t + SR^t\times(-NM_1^t)$$
$$= -NM_1^t\ .$$

The second part of (ii), and (iii) follow in a similar way.

This completes the proof .

CHAPTER 6

GOLAY SEQUENCES AND OTHER SEQUENCES WITH ZERO AUTO-CORRELATION FUNCTION

In this chapter we consider sequences with zero non-periodic auto-correlation function and show how such sequences may be used to construct orthogonal designs. The results of this chapter arise from joint work carried out with Dr Jennifer Seberry Wallis.

6.1 Preliminaries

Let $X = \{A_1, A_2, \ldots, A_m\}$, where $A_i = \{a_{i1}, \ldots, a_{in}\}$, be m sequences of commuting variables of length n.

The non-periodic auto-correlation function of the family of sequences X (denoted N_X) is a function defined by

$$N_X(j) = \sum_{i=1}^{n-j} \left(a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j}\right).$$

The periodic auto-correlation function of the family of sequences X (denoted P_X) is defined by

$$P_X(j) = \sum_{i=1}^{n} \left(a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j}\right)$$

where we assume the second subscript is actually chosen from the complete set of residues $\mathrm{mod}(n)$.

We can interpret the function P_X in the following way: form the m circulant matrices which have first rows respectively,

$$|a_{11}\ a_{12} \ldots a_{1n}|,\ |a_{21}\ a_{22} \ldots a_{2n}|,\ \ldots,\ |a_{m1}\ a_{m2} \ldots a_{mn}|,$$

then $P_X(j)$ is the sum of the inner products of rows 1 and $j+1$ of these matrices.

Clearly if X is a family of sequences as above, then

$$P_X(j) = N_X(j) + N_X(n-j),\quad j = 1, \ldots, n-1,$$

and

$$N_X(j) = 0 \ \forall j \Rightarrow P_X(j) = 0 \ \forall j .$$

Note. $P_X(j)$ may equal 0 for all $j = 1, \ldots, n-1$ even though the $N_X(j)$ are not.

We say the *weight* of X is the number of non-zero entries in X.

Let X be as above with $N_X(j) = 0 \ \forall j = 1, 2, \ldots, n-1$ then we will call X a set of *m-complementary sequences* of length n.

If $X = \{A_1, A_2, \ldots, A_m\}$ are m-complementary sequences of length n and weight $2k$ such that

$$Y = \{(A_1+A_2)/2, (A_1-A_2)/2, \ldots, (A_{2i-1}+A_{2i})/2, (A_{2i-1}-A_{2i})/2, \ldots\}$$

are also m-complementary sequences (of weight k) then X will be said to be a set of *m-complementary disjointable sequences* of length n. X will be said to be a set of *m-complementary disjoint sequences* of length n if all $\binom{m}{2}$ pairs of sequences are disjoint; that is, $A_i * A_j = 0 \ \forall i, j$ where $*$ is the Hadamard product.

One more piece of notation is in order. If g_r denotes a sequence of integers of length r then by xg_r we mean the sequence of length r obtained from g_r by multiplying each member of g_r by x.

If $X = \{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}\}$ ~~are~~ is a set of two sequences where $a_i, b_j \in \{1, -1\}$ and $N_X(j) = 0$ for $j = 1, \ldots, n-1$ then the sequences in X are called *Golay complementary sequences of length* n. For example

$n = 2$ 1 1 and 1 -

$n = 10$ 1 - - 1 - 1 - - - 1 and 1 - - - - - - 1 1 -

$n = 26$ 1 1 1 - - 1 1 1 - 1 - - - - - 1 - 1 1 - - 1 - - - - and

- - - 1 1 - - - 1 - 1 1 - 1 - 1 - 1 1 - - 1 - - - - .

6.2 Constructing sequences

We will use the notation A^* to mean the order of the entries in the sequence A are reversed.

Then a few simple observations are in order and for convenience we put them together as a lemma.

LEMMA 6.2.1. *Let* $X = \{A_1, A_2, \ldots, A_m\}$ *be a set of* m*-complementary sequences of length* n . *Then*

(*i*)
$$Y = \{A_1^*, A_2^*, \ldots, A_i^*, A_{i+1}, \ldots, A_m\}$$
is a set of m*-complementary sequences of length* n ;

(*ii*)
$$W = \{A_1, \ldots, A_i, -A_{i+1}, \ldots, -A_m\}$$
is a set of m*-complementary sequences of length* n ;

(*iii*) *if* m *is even, then*
$$Z = \{\{A_1, A_2\}, \{A_1, -A_2\}, \ldots, \{A_{m-1}, A_m\}, \{A_{m-1}, -A_m\}\}$$
is a set of m*-complementary sequences of length* $2n$.

If m *is odd, let* A_{m+1} *be a sequence of* n *zeros. Then*
$$Z = \{\{A_1, A_2\}, \{A_1, -A_2\}, \ldots, \{A_m, A_{m+1}\}, \{A_m, -A_{m+1}\}\}$$
is a set of $m+1$*-complementary sequences of length* $2n$.

(*iv*) *If* m *is even, then*
$$U = \{\{A_1/A_2\}, \{A_1/-A_2\}, \ldots, \{A_{m-1}/A_m\}, \{A_{m-1}/-A_m\}\} ,$$
where A_j/A_k *means* $a_{j1}, a_{k1}, a_{j2}, a_{k2}, \ldots, a_{jn}, a_{kn}$,
is a set of m*-complementary sequences of length* $2n$.

If m *is odd, let* A_{m+1} *be the sequence of* n *zeros. Then*
$$U = \{\{A_1/A_2\}, \{A_1/-A_2\}, \ldots, \{A_m/A_{m+1}\}, \{A_m/-A_{m+1}\}\}$$
is a set of $m+1$*-complementary sequences of length* $2n$.

THEOREM 6.2.2. *Suppose* $X = \{A_1, \ldots, A_{2m}\}$ ~~*are*~~ is a set of $2m$*-complementary sequences of length* n *and weight* l *and* $Y = \{B_1, B_2\}$ ~~*are*~~ is a set of *2-complementary disjointable sequences of length* t *and weight* $2k$. *Then there are* $2m$*-complementary sequences of length* nt *and weight* kl .

The same result is true if X ~~*are*~~ is a set of $2m$*-complementary disjointable sequences of length* n *and weight* $2l$ *and* Y ~~*are*~~ is a set of *2-complementary* *sequences of weight* k .

Proof. Using an idea of R.J. Turyn we consider

$$A_{2i-1} \times (B_1^*+B_2^*)/2 + A_{2i} \times (B_1^*-B_2^*)/2$$

and

$$A_{2i-1} \times (B_1-B_2)/2 - A_{2i} \times (B_1+B_2)/2$$

for $i = 1, \ldots, m$, which are the required sequences in the first case, and

$$(A_{2i-1}+A_{2i})/2 \times B_1 + (A_{2i-1}-A_{2i})/2 \times B_2^*$$

and

$$(A_{2i-1}+A_{2i})/2 \times B_2 - (A_{2i-1}-A_{2i})/2 \times B_1^*$$

for $i = 1, \ldots, m$, which are the required sequences for the second case.

COROLLARY 6.2.3. *Since there are Golay sequences of lengths* 2, 10 *and* 26 *there are Golay sequences of length* $2^a 10^b 26^c$ *for* a, b, c *non-negative integers.*

COROLLARY 6.2.4. *There are* 2*-complementary sequences of lengths* $2^a 6^b 10^c 14^d 26^e$ *of weights* $2^a 5^b 10^c 13^d 26^e 2$ *where* a, b, c, d, e *are non-negative integers.*

Proof. We use Golay sequences of lengths 2, 10 and 26 and the following 2-complementary disjointable sequences of lengths 6 and 14 respectively: $\{\{1\ 1\ -\ 1\ 0\ 1\}, \{1\ 1\ -\ -\ 0\ -\}\}$ and $\{\{1\ 1\ 1\ -\ -\ 1\ 1\ 1\ -\ 1\ -\ -\ 0\ -\}, \{1\ 1\ 1\ 1\ -\ 1\ 1\ -\ -\ 1\ -\ 1\ 0\ 1\}\}$.

LEMMA 6.2.5. *Consider four* (1, -1) *sequences* $A = \{X, Y, Z, W\}$ *where*

$$X = \{x_1 = 1, x_2, x_3, \ldots, x_m, h_m x_m, \ldots, h_3 x_3, h_2 x_2, h_1 x_1 = -1\} ,$$

$$U = \{u_1 = 1, u_2, u_3, \ldots, u_m, f_m u_m, \ldots, f_3 u_3, f_2 u_2, f_1 u_1 = 1\} ,$$

$$Y = \{y_1, y_2, \ldots, y_{m-1}, y_m, g_{m-1} y_{m-1}, \ldots, g_3 y_3, g_2 y_2, g_1 y_1\} ,$$

$$V = \{v_1, v_2, \ldots, v_{m-1}, v_m, e_{m-1} v_{m-1}, \ldots, e_3 v_3, e_2 v_2, e_1 v_1\} .$$

Then $N_A = 0$ *implies that* $h_i = f_i$, $i > 1$, *and* $g_j = e_j$. *Here*

$$8m - 2 = \left(\sum_{i=1}^{m} x_i + x_i h_i\right)^2 + \left(\sum_{i=1}^{m} u_i + u_i f_i\right)^2 + \left(y_m + \sum_{i=1}^{m-1} y_i + y_i g_i\right)^2 + \left(v_m + \sum_{i=1}^{m-1} v_i + v_i e_i\right)^2 .$$

Proof. We note since all variables are ± 1 , $a + b \equiv ab + 1 \pmod 4$ and $x + xyz \equiv y + z \pmod 4$. Clearly $N_A(2m-1) = 0$ gives $-h_1 = f_1 = 1$, and

$$\begin{aligned} N_A(2m-2) &= x_1 x_2 h_2 + x_2 h_1 x_1 + f_2 u_1 u_2 + u_2 f_1 u_1 + g_1 y_1^2 + e_1 v_1^2 \\ &\equiv h_1 + h_2 + f_1 + f_2 + g_1 + e_1 \pmod 4 \\ &\equiv h_2 f_2 + g_1 e_1 + 2 \pmod 4 \\ &\equiv 0 \pmod 4 . \end{aligned}$$

This gives $h_2 f_2 = g_1 e_1$. We proceed by induction to show that $h_i f_i = g_{i-1} e_{i-1}$ for all $i \le m$.

Assume $h_i f_i = g_{i-1} e_{i-1}$, that is, $h_i + f_i + g_{i-1} + e_{i-1} \equiv 0 \pmod 4$, for all $i < k \le m$. Now consider

$$\begin{aligned} N_A(2m-k) &= \left(x_1 x_k h_k + x_2 x_{k-1} h_{k-1} + \dots + x_{k-1} x_2 h_2 + x_k x_1 h_1\right) \\ &\quad + \left(u_1 u_k f_k + u_2 u_{k-1} f_{k-1} + \dots + u_{k-1} u_2 f_2 + u_k u_1 f_1\right) \\ &\quad + \left(y_1 y_{k-1} g_{k-1} + y_2 y_{k-2} g_{k-2} + \dots + y_{k-2} y_2 g_2 + y_{k-1} y_1 g_1\right) \\ &\quad + \left(v_1 v_{k-1} e_{k-1} + v_2 v_{k-2} e_{k-2} + \dots + v_{k-2} v_2 e_2 + v_{k-1} v_1 e_1\right) \\ &\equiv h_1 + \dots + h_k + f_1 + \dots + f_k + g_1 + \dots + \\ &\qquad + g_{k-1} + e_1 + \dots + e_{k-1} \pmod 4 \\ &\equiv h_k f_k + g_{k-1} e_{k-1} + 2 \pmod 4 \\ &\equiv 0 \pmod 4 . \end{aligned}$$

This gives the result for all $1 < i \le m$.

Suppose $k = m + j < 2m$. Then

$$
\begin{aligned}
N_A(2m-k) = {} & \left(x_1 x_{m-j+1} + \dots + x_j x_m\right) + \left(x_{j+1} h_m x_m + \dots + x_m h_{j+1} x_{j+1}\right) \\
& + \left(h_m x_m h_j x_j + \dots + h_{m-j+1} x_{m-j+1} h_1 x_1\right) \\
& + \left(u_1 u_{m-j+1} + \dots + u_j u_m\right) + \left(u_{j+1} f_m u_m + \dots + u_m f_{j+1} u_{j+1}\right) \\
& + \left(f_m u_m f_j u_j + \dots + f_{m-j+1} u_{m-j+1} f_1 u_1\right) \\
& + \left(y_1 y_{m-j+1} + \dots + y_j y_m\right) + \left(y_{j+1} g_{m-1} y_{m-1} + \dots + y_m g_j y_j\right) \\
& + \left(g_{m-1} y_{m-1} g_{j-1} y_{j-1} + \dots + g_{m-j+1} y_{m-j+1} g_1 y_1\right) \\
& + \left(v_1 v_{m-j+1} + \dots + v_j m_j\right) + \left(v_{j+1} e_{m-1} v_{m-1} + \dots + v_m e_j v_j\right) \\
& + \left(e_{m-1} v_{m-1} e_{j-1} v_{j-1} + \dots + e_{m-j+1} v_{m-j+1} e_1 v_1\right) \\
\equiv {} & -\left(h_1 + \dots + h_m + h_{m-j+1} + \dots + h_m\right) \\
& + \left(f_1 + \dots + f_m + f_{m-j+1} + \dots + f_m\right) \\
& + \left(g_1 + \dots + g_{m-1} + g_{m-j+1} + \dots + g_{m-1} + 1\right) \\
& + \left(e_1 + \dots + e_{m-1} + e_{m-j+1} + \dots + e_{m-1} + 1\right) \pmod 4 \\
\equiv {} & h_1 f_1 + h_{m-j+1} f_{m-j+1} \pmod 4 \\
\equiv {} & h_{m-j+1} f_{m-j+1} - 1 \pmod 4 .
\end{aligned}
$$

Hence $h_{m-j+1} f_{m-j+1} = 1$. So in general $h_i f_i = 1$, $i > 1$, and $e_i g_i = 1$.

By constructing the circulant matrix with X as its first row, we can see that $\left(\sum_{i=1}^{m} x_i + x_i h_i\right)^2$ is the sum of the innerproducts of row 1 with rows 1, 2, ..., m ; the innerproduct of row 1 with itself is the sum of the squares of the elements of X .

By repeating this argument with U, Y and V and noting that $N_A = 0$ implies $P_A = 0$, we can see that

$$
8m - 2 = \left(\sum_{i=1}^{m} x_i + x_i h_i\right)^2 + \left(\sum_{i=1}^{m} u_i + u_i f_i\right)^2 + \left(y_m + \sum_{i=1}^{m-1} y_i + y_i g_i\right)^2 + \left(v_m + \sum_{i=1}^{m-1} v_i + e_i v_i\right)^2 .
$$

COROLLARY 6.2.6. *Consider four* $(1, -1)$ *sequences* $A = \{X, U, Y, W\}$ *where*

$$X = \{x_1 = 1, x_2, x_3, \ldots, x_m, -x_m, \ldots, -x_3, -x_2, -x_1 = -1\} ,$$

$$U = \{u_1 = 1, u_2, u_3, \ldots, u_m, f_m u_m, \ldots, f_3 u_3, f_2 u_2, f_1 u_1 = 1\} ,$$

$$Y = \{y_1, y_2, \ldots, y_{m-1}, y_m, y_{m-1}, \ldots, y_3, y_2, y_1\} ,$$

$$V = \{v_1, v_2, \ldots, v_{m-1}, v_m, e_{m-1}v_{m-1}, \ldots, e_3 v_3, e_2 v_2, e_1 v_1\} .$$

Then $N_A = 0$ *implies that all* e_i *are* $+1$ *and all* f_i *,* $i > 1$ *, are* -1 *. Here* $8m - 6$ *is the sum of two squares.*

Similarly we can prove

COROLLARY 6.2.7. *Consider four* $(1, -1)$ *sequences* $A = \{X, U, Y, W\}$ *where*

$$X = \{x_1 = 1, x_2, x_3, \ldots, x_m, x_{m+1}, x_m, \ldots, x_3, x_2, x = 1\} ,$$

$$U = \{u_1 = 1, u_2, u_3, \ldots, u_m, u_{m+1}, f_m u_m, \ldots, f_3 x_3, f_2 x_2, -1\} ,$$

$$Y = \{y_1, y_2, \ldots, y_m, -y_m, \ldots, -y_2, -y_1\} ,$$

$$V = \{v_1, v_2, \ldots, v_m, e_m v_m, \ldots, e_2 v_2, e_1 v_1\} ,$$

which have $N_A = 0$ *. Hence* $e_i = -1$ *for all* i *and* $f_i = +1$ *for all* i *. Here* $8m + 2$ *is the sum of two squares.*

DEFINITION 6.2.8. Sequences such as those described in the last two corollaries will be called *Turyn sequences* of length l (the four sequences are of weights $l, l, l - 1$ and $l - 1$). A sequence $X = \{x_1, \ldots, x_n\}$ will be called *skew* if n is even and $x_i = -x_{n-i+1}$ and *symmetric* if n is odd and $x_i = x_{n-i+1}$.

LEMMA 6.2.9. *There exist Turyn sequences of lengths* 2, 4, 6, 8, 3, 5, 7, 13 *and* 15 .

Proof. Consider

$l = 2$: $X = \{\{1-\}, \{11\}, \{1\}, \{1\}\}$,

$l = 4$: X = {{11--}, {11-1}, {111}, {1-1}} ,

$l = 6$: X = {{111---}, {11-1-1}, {11-11}, {11-11}} ,

$l = 8$: X = {{11-1-1--}, {1111---1}, {111-111}, {1--1--1}} ,

$l = 3$: X = {{111}, {11-}, {1-}, {1-}} ,

$l = 5$: X = {{11-11}, {1111-}, {11--}, {1-1-}} ,

$l = 7$: X = {{111-111}, {11---1-}, {11-1--}, {11-1--}} ,

$l = 13$: X = {{1111-1-1-1111}, {111--1-1--11-}, {111-11--1---},
{111--1-11---}} ,

or

X = {{111-11-11-111}, {111--1-1--11-}, {11---1-111--},
{1111-1-1----}} ,

$l = 15$: X = {{11-111-1-111-11}, {111-11---11-11-}, {1111--1-11----},
{1----1-1-1111-}} .

Remark. These sequences were constructed using the Research School of Physical Sciences DEC-10 System.

A complete computer search gave no more solutions for l even and less than 20 , or l odd and less than 30 .

6.3 Using sequences to construct orthogonal designs

Before we proceed with the first result, we need one simple remark. If we have a collection of sequences, X (each of length n), such that $N_X(j) = 0$, $j = 1, \ldots, n-1$, then we may augment each sequence at the beginning with k zeros and at the end with l zeros so that the resulting collection, say X_1, of sequences having length $n + k + l$ still has $N_{X_1}(j) = 0$, $j = 1, \ldots, n+k+l-1$. We use this remark in the next result.

THEOREM 6.3.1. *Let* r_1 *be any number of the form* $2^{a_1}10^{b_1}26^{c_1}5^{d_1}13^{e_1}$ *and let* $\bar{r}_2$ *be any number of the form* $2^{a_2}10^{b_2}26^{c_2}5^{d_2}13^{e_2}$, a_i, b_i, c_i, d_i *and* e_i *non-negative integers. Further, let* n *be any integer greater than or equal to* $\max\left(2^{a_1}10^{b_1}26^{c_1}6^{d_1}14^{e_1}, 2^{a_2}10^{b_2}26^{c_2}6^{d_2}14^{e_2}\right)$. *Then there exist orthogonal designs of order* $4n$ *and types*

(i) (r_1, r_1, r_2, r_2) *and*

(ii) $(1, 4, r_i, r_i)$, $i = 1, 2$.

Proof. If we have 4-complementary sequences of length n, then we may use these sequences as first rows of circulant matrices which satisfy the conditions of Theorem 1.5.

We note that if $\{X, Y\}$ ~~are~~ is a set of 2-complementary sequences then $\{aX/bY, bX/\bar{a}Y\}$ ~~are~~ is a set of 2-complementary sequences on the variables a and b.

Now, by appealing to Corollary 6.2.4, we obtain (i) and by using the sequences $\{cd\bar{c}, c0c\}$, with zeros adjoined, we obtain (ii).

Before we proceed, we quote a theorem of Cooper and Wallis [2].

THEOREM 6.3.2. *Suppose there exists four circulant* $(0, 1, -1)$ *matrices* X_i, $i = 1, 2, 3, 4$ *which are non-zero for each of the* n^2 *entries for only one* i *and which satisfies*

$$\sum_{i=1}^{4} X_i X_i^t = xI \ .$$

Then there is a Baumert-Hall array of order n .

Note. A *Baumert-Hall array* for order n is an orthogonal design of order $4n$ and type (n, n, n, n) .

In order to find circulant matrices which satisfy the conditions of this theorem we are led to the following definition:

DEFINITION 6.3.3. 4-complementary disjoint $(0, 1, -1)$ sequences of length t and total weight t will be called *T-sequences*.

EXAMPLE 6.3.4. Consider

$$T = \{100000, 011000, 0001\text{-}0, 000001\} \ .$$

The sequences are disjoint as the ith entry is non-zero in one and only one of the four sequences. The total weight is 6 and $N_T = 0$.

We now give a way of obtaining T-sequences from Turyn sequences.

THEOREM 6.3.5 (Turyn). *Suppose* $A = \{X, U, Y, V\}$ *are Turyn sequences of length* l *where* X *is skew and* Y *is symmetric for* l *even and* X *is symmetric and* Y *is skew for* l *odd. Then there are* T*-sequences of length* $2l - 1$ *and* $4l - 1$.

Proof. We use the notation A/B as before to denote the interleaving of two sequences $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_{m-1}\}$:

$$A/B = \{a_1, b_1, a_2, b_2, \ldots, b_{m-1}, a_m\} \ .$$

Let 0_t be a sequence of zeros of length t . Then

$$T_1 = \{\{\tfrac{1}{2}(X+U), 0_{l-1}\}, \{\tfrac{1}{2}(X-U), 0_{l-1}\}, \{0_l, \tfrac{1}{2}(Y+V)\}, \{0_l, \tfrac{1}{2}(Y-V)\}\}$$

and

$$T_2 = \{\{1, 0_{4l-2}\}, \{0, X/Y, 0_{2l-1}\}, \{0, 0_{2l-1}, U/0_{l-1}\}, \{0, 0_{2l-1}, 0_l/V\}\}$$

are the T-sequences of lengths $2l - 1$ and $4l - 1$ respectively.

COROLLARY 6.3.6. *There are* T*-sequences constructed from Turyn sequences of lengths* 3, 5, 7, 9, 11, 13, 15, 19, 23, 25, 27, 29, 31, 51, 59 .

We now use these T-sequences as first rows of circulant matrices. By using Theorem 6.3.2, we obtain

LEMMA 6.3.7. *There are orthogonal designs of order* $4n$ *and type* (n, n, n, n) *for*

$n = 3, 5, 7, 9, 11, 13, 15, 19, 23, 25, 27, 29, 31, 51$ *and* 59 .

REFERENCES

[1] L.D. Baumert and Marshall Hall, Jr., "A new construction for Hadamard matrices", *Bull. Amer. Math. Soc.* 71 (1965), 169-170.

[2] Joan Cooper and Jennifer Wallis, "A construction for Hadamard arrays", *Bull. Austral. Math. Soc.* 7 (1972), 269-278.

[3] Peter Eades, "Orthogonal designs constructed from circulants", *Utilitas Math.* (to appear).

[4] Peter Eades, "Some asymptotic existence results for orthogonal designs", *Ars Combinatoria* 1 (1976), 109-118.

[5] Peter Eades, "Some new constructions for orthogonal designs using circulants", *Combinatorics V: Proc. Fifth Australian Conf*, Lecture Notes in Mathematics, . Springer-Verlag, Berlin, Heidelberg, New York, (to appear)

[6] Peter Eades and Richard M. Hain, "On circulant weighing matrices", submitted.

[7] Anthony V. Geramita, Joan Murphy Geramita and Jennifer Seberry Wallis, "Orthogonal designs", *Linear and Multilinear Algebra* 3 (1975/76), 281-306.

[8] A.V. Geramita and N.J. Pullman, "Radon's function and Hadamard arrays", *Linear and Multilinear Algebra* 2 (1974), 147-150.

[9] A.V. Geramita and J.H. Verner, "Orthogonal designs with zero diagonal", *Canad. J. Math.* 28 (1976), 215-224.

[10] Anthony V. Geramita and Jennifer Seberry Wallis, "Some new constructions for orthogonal designs", *Combinatorial Mathematics IV: Proc. Fourth Australian Conf.*, Lecture Notes in Mathematics, Vol 560(1976), 46-54, Springer-Verlag, Berlin, Heidelberg, New York, .

[11] J.M. Goethals and J.J. Seidel, "A skew-Hadamard matrix of order 36 ", *J. Austral. Math. Soc.* 11 (1970), 343-344.

[12] M.J.E. Golay, "Static multislit spectrometry and its applications to the panoramic display of infrared spectra", *J. Opt. Soc. Amer.* 41 (1951), 468-472.

[13] M.J.E. Golay, "Complementary series", *IRE Trans. Information Theory* IT- 7 (1961), 82-87.

[14] M.J.E. Golay, "Note on complementary series", *Proc. IRE* 50 (1962), 84.

[15] Marshall Hall Jr., *Combinatorial Theory*, Blaisdell, Waltham, Mass., 1967.

[15½] Radon, J., "Linear Scharen Orthogonalen Matrizen", Abh. Math. Sem. Univ. Hamburg 1(1922) 1-14.

[16] Peter J. Robinson, "A non-existence theorem for orthogonal designs", *Utilitas Math.* 10 (1976),

[17] Peter J. Robinson, "Amicable orthogonal designs", *Bull. Austral. Math. Soc.* 14 (1976), 303-314.

[18] Peter J. Robinson, "Orthogonal designs in order 24 ", *Combinatorics V: Proc. Fifth Australian Conf.*, Lecture Notes in Mathematics, . Springer-Verlag, Berlin, Heidelberg, New York, (to appear).

[19] D. Shapiro, "Similarities, quadratic forms, and Clifford algebras", PhD Dissertation, University of California, Berkeley, 1974.

[20] D. Shapiro, Private Communication, 1975.

[21] Richard J. Turyn, "Four-phase Barker codes", *IEEE Trans. Information Theory*, IT-20 (1974), 366-371.

[22] Richard J. Turyn, "Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression and surface wave encodings", *J. Combinatorial Theory Ser. A* 16 (1974), 313-333.

[23] Jennifer Seberry Wallis, "On the existence of Hadamard matrices", *J. Combinatorial Theory Ser. A* 21 (1976), 444-451.

[24] Jennifer Seberry Wallis, "Orthogonal designs V: Orders divisible by eight", *Utilitas Math.* 9 (1976), 263-281.

[25] John Williamson, "Hadamard's determinant theorem and the sum of four squares", *Duke Math. J.* 11 (1944), 65-81.

[26] Warren W. Wolfe, "Rational quadratic forms and orthogonal designs", *Queen's Mathematical Preprint* No. 1975-22.

[27] Warren W. Wolfe, "Orthogonal designs - amicable orthogonal designs - some algebraic and combinatorial techniques", PhD Dissertation, Queen's University, Kingston, Ontario, 1975.

[28] Warren W. Wolfe, "Clifford algebras and amicable orthogonal designs", *Queen's Mathematical Preprint* No. 1975-22.

APPENDIX I

PRODUCT DESIGNS IN POWERS OF TWO

Before we give some product designs, and their constructions, we note the following:

1. There are amicable orthogonal designs of order 2^t and type $\left((1, 1, 2, 4, \ldots, 2^{t-1}); (2^{t-2}, 2^{t-2}, 2^{t-1})\right)$.
2. Lemma 4.3.5 gives the following amicable orthogonal designs of order 8 :

   $\left((1, 1, 2, 2, 2); (8)\right)$ $\qquad$ $\left((1, 7); (1, 7)\right)$

   $\left((1, 2, 2, 3); (2, 6)\right)$
3. By appealing to Corollary 4.3.3, and using the amicable orthogonal designs of order 8 and type $\left((1, 7); (1, 7)\right)$, we obtain amicable orthogonal designs of order 32 and type $\left((1, 1, 2, 28); (4, 28)\right)$.
4. If $(M_1; M_2; N)$ are product designs, then $(M_2; M_1; N)$ are product designs.
5. Variables in product designs or amicable designs may be equated.

In Tables 1, 2, 3 and 4 we give product designs of orders 4, 8, 16, 32 and 64 together with their construction.

Unless otherwise stated, the constructions used are those of Theorem 5.2.6 and Corollary 5.2.4.

We list those obtained from Theorem 5.2.6 first.

PRODUCT DESIGNS OF ORDERS 4 AND 8

Product designs	Construction	
order 4		
(1,1,1;1,1,1;1)	Construction 5.1.1	
order 8		
(1,1,1,2;1,1,3;3)	((1,1,2);(1,3))	((1,1);(1,1))
(1,1,2,3;1,3,3;1)	((1,1,2);(3,1))	((1,1);(1,1))
(1,1,2,2;2,2,2;2)	((1,1,2);(2,2))	((1,1);(1,1))
(1,1,1,1;2,2;2,2)	(1,1,1;1,2;1)	
(1,1,1,2;1,4;1,2)	(1,1,1;2,1;1)	
(1,1,1;1,1,1;5)	Example 5.2.1	

TABLE 1

PRODUCT DESIGNS OF ORDER 16

Product designs	Construction	
(1,1,2,3;1,1,2,3;9)	[(1,1,2);(1,3)]	[(1,1,2);(1,3)]
(1,1,2,9;1,3,3,6;3)	[(1,1,2);(3,1)]	[(1,1,2);(1,3)]
(1,1,7;1,1,7;7)	[(1,7);(1,7)]	[(1,1);(1,1)]
(1,2,2,2,3;2,2,6;6)	[(1,2,2,3);(2,6)]	[(1,1);(1,1)]
(2,2,2,4;1,1,2,6;6)	[(1,1,2);(1,3)]	[(1,1,2);(2,2)]
(2,2,4,5;3,5,5;3)	[(2,2,4);(5,3)]	[(1,1);(1,1)]
(2,2,4,6;2,3,3,6;2)	[(1,1,2);(3,1)]	[(1,1,2);(2,2)]
(2,2,4,7;1,7,7;1)	[(2,2,4);(7,1)]	[(1,1);(1,1)]
(2,6,6;1,1,2,4,6;2)	[(1,1);(1,1)]	[(1,1,2,4);(2,6)]
(4,4,4;1,1,2,4,4;4)	[(1,1);(1,1)]	[(1,1,2,4);(4,4)]
(1,1,1,1,2;2,4;2,4,4)	(1,1,1,1;2,2;2,2)	
(1,1,1,2,2;3,4;3,6)	(1,1,1,2;2,3;3)	
(1,1,1,2,3;2,6;2,6)	(1,1,2,3;1,6;1)	
(1,1,1,2,4;1,8;1,2,4)	(1,1,1,2;4,1;1,2)	
(1,1,1,2;5;3,3,5)	(1,1,1,2;5;3) in Theorem 5.2.5	
(1,1,2,3,4;3,8;2,3)	(1,1,2,3;4,3;1)	
(1,1,2,3,6;1,12;1,2)	(1,1,2,3;6,1;1)	
(1,2,2;2,3;3,4,4)	(2,2;1,3;2,2)	
(1,2,3,3;4,5;2,5)	(1,3,3;2,5;1)	
(2,2,2,5;1,10;1,4)	(2,2,2;5,1;2)	

TABLE 2

PRODUCT DESIGNS OF ORDER 32

Product designs	Construction	
(3,13,13;4,4,8,13;3)	$\big((1,1);(1,1)\big)$	$\big((4,4,8);(3,13)\big)$
(3,5,5,10;2,2,4,15;9)	$\big((1,1,2);(1,3)\big)$	$\big((2,2,4);(5,3)\big)$
(4,4,8,12;3,3,4,6,12;4)	$\big((1,1,2);(3,1)\big)$	$\big((1,1,2,4);(4,4)\big)$
(7,7,9;4,4,7,8;9)	$\big((1,1);(1,1)\big)$	$\big((4,4,8);(7,9)\big)$
(1,1,1,1,2,4;2,8;2,4,8,8)	(1,1,1,1,2;4,2;2,4,4)	
(1,1,1,2,2,4;3,8;3,6,12)	(1,1,1,2,2;4,3;3,6)	
(1,1,1,2,4,8;1,16;1,2,4,8)	(1,1,1,2,4;8,1;1,2,4)	
(1,1,2,2,3;4,5;5,18)	(1,1,2,3;2,5;9)	
(1,1,2,3,4,11;22;4,6)	(1,1,2,3,4;11;2,3)	
(1,1,2,3,4,8;3,16;3,4,6)	(1,1,2,3,4;8,3;2,3)	
(1,1,2,3,6,12;1,24;1,2,4)	(1,1,2,3,6;12,1;1,2)	
(1,1,2,6,9;7,12;6,7)	(1,1,2,9;6,7;3)	
(1,1,2,3,9;6,10;6,10)	(1,1,2,9;3,10;3)	
(1,1,2,9,10;3,20;3,6)	(1,1,2,9;10,3;3)	
(1,2,3,3,4;5,8;4,5,10)	(1,2,3,3;4,5;2,5)	
(1,2,3,3,6;4,11;6,11)	(1,3,3,6;2,11;3)	
(1,3,3,6,10;3,20;3,6)	(1,3,3,6;10,3;3)	
(1,3,3,6,11;2,22;2,6)	(1,3,3,6;11,2;3)	
(1,3,3,6,9;4,18;4,6)	(1,3,3,6;9,4;3)	
(1,4,7,7;8,11;2,11)	(1,7,7;4,11;1)	
(1,6,7,7;9,12;2,9)	(1,7,7;6,9;1)	
(2,2,2,4,9;1,18;1,12)	(2,2,2,4;9,1;6)	
(2,2,2,5,10;1,20;1,2,8)	(2,2,2,5;10,1;1,4)	
(2,2,4,5,10;3,20;3,6)	(2,2,4,5;10,3;3)	
(2,2,4,5,8;5,16;5,6)	(2,2,4,5;8,5;3)	
(2,2,4,6,11;3,22;3,4)	(2,2,4,6;11,3;2)	
(2,2,4,6,9;5,18;4,5)	(2,2,4,6;9,5;2)	
(2,2,4,7,8;7,16;2,7)	(2,2,4,7;8,7;1)	
(2,6,6,13;1,26;1,4)	(2,6,6;13,1;2)	
(4,4,4,7;5,14;5,8)	(4,4,4;7,5;4)	

TABLE 3

PRODUCT DESIGNS OF ORDER 64

Product Designs	Construction
(1,1,1,1,2,2,4;4,8;4,8,8,16,16)	(1,1,1,1,2,4;2,8;2,4,8,8)
(1,1,1,2,2,3,4;6,8;6,8,12,24)	(1,1,1,2,2,4;3,8;3,6,12)
(1,1,1,2,4,8,16;1,32;1,2,4,8,16)	(1,1,1,2,4,8;16,1;1,2,4,8)
(1,1,2,2,3,4;5,8;5,10,36)	(1,1,2,2,3;4,5;5,18)
(1,1,2,3,4,8,16;3,32;3,6,8,12)	(1,1,2,3,4,8;16,3;3,4,6)
(1,1,2,3,6,9;10,12;10,12,20)	(1,1,2,3,9;6,10;6,10)
(1,1,2,6,7,9;12,14;12,12,14)	(1,1,2,6,9;7,12;6,7)
(1,2,3,3,4,5;8,10;8,8,10,20)	(1,2,3,3,4;5,8;4,5,10)
(1,2,3,3,6,11;4,22;4,12,22)	(1,2,3,3,6;11,4;6,11)
(1,3,3,6,9,18;4,36;4,8,12)	(1,3,3,6,9;18,4;4,6)
(1,4,7,7,8;11,16;4,11,22)	(1,4,7,7;8,11;2,11)
(1,6,7,7,12;9,24;4,9,18)	(1,6,7,7;12,9;2,9)
(2,2,2,5,10,20;1,40;1,2,4,16)	(2,2,2,5,10;20,1;1,2,8)
(2,2,4,6,9,18;5,36;5,8,10)	(2,2,4,6,9;18,5;4,5)
(2,2,4,7,8,16;7,32;7,18)	(2,2,4,7,8;16,7;9)
(2,3,5,5,10;4,21;18,21)	(3,5,5,10;2,21;9)
(2,6,6,13,26;1,52;1,2,8)	(2,6,6,13;26,1;1,4)
(3,4,13,13;8,25;6,25)	(3,13,13;4,25;3)
(4,4,4,7,14;5,28;5,10,16)	(4,4,4,7;14,5;5,8)
(4,4,7,8,12;14,21;8,21)	(4,4,8,12;7,21;4)
(7,7,9,12;11,24;11,18)	(7,7,9;12,11;9)

TABLE 4

APPENDIX II

ORTHOGONAL DESIGNS IN POWERS OF TWO

In Tables 5, 6, 7, 8, 9, 10, 11 and 12 we give orthogonal designs of orders 32, 64 and 128 .

These designs are obtained by using product designs given in Appendix I and appealing to Theorem 5.3.1.

10 VARIABLE ORTHOGONAL DESIGNS OF ORDER 32

Orthogonal design	Construction	
	Product designs	Amicable designs
(1,1,1,1,2,2,3,3,9,9)	(1,1,2,3;1,3,3;1)	((1,3);(1,1,2))
(1,1,1,1,2,2,4,4,8,8)	(1,1,1,1,2;2,4;2,4,4)	((1,1);(1,1))
(1,1,1,2,3,3,3,3,6,9)	(1,1,1,2;1,1,3;3)	((1,3);(1,1,2))
(1,1,3,3,3,3,3,3,6,6)	(1,1,1,2;1,1,3;3)	((3,1);(1,1,2))
(1,2,2,2,2,2,3,6,6,6)	(1,2,2,2,3;2,2,6;6)	((1,1);(1,1))

TABLE 5

9 VARIABLE ORTHOGONAL DESIGNS OF ORDER 32

Orthogonal design	Construction	
	Product designs	Amicable designs
(1,1,1,1,2,2,3,7,14)	(1,1,2,3;7;1)	((1,1,2);(1,1,2))
(1,1,1,1,2,2,4,10,10)	(1,1,1,1,2;2,4;10)	((1,1);(1,1))
(1,1,1,1,2,4,7,7,8)	(1,1,1,2,4;1,8;7)	((1,1);(1,1))
(1,1,1,1,4,4,6,6,8)	(1,1,1,1;2,2;4)	((1,3);(1,1,2))
(1,1,1,2,2,3,4,6,12)	(1,1,1,2,2;3,4;3,6)	((1,1);(2))
(1,1,1,2,2,3,6,8,8)	(1,1,1,2,3;2,6;8)	((1,1);(1,1))
(1,1,1,2,2,4,7,7,7)	(1,1,1;1,1,1;1)	((1,7);(2,2,4))
(1,1,1,3,3,3,5,5,10)	(1,1,1;1,1,1;5)	((1,3);(1,1,2))
(1,1,1,2,3,3,5,6,10)	(1,1,1,2;5;3)	((1,1,2);(1,1,2))
(1,1,2,2,2,4,6,7,7)	(1,1,2,3;7;1)	((2,1,1);(1,1,2))
(1,1,2,3,3,4,4,6,8)	(1,1,2,3,4;3,8;2,3)	((1,1);(2))
(1,1,2,3,3,4,5,5,8)	(1,1,2,3,4;3,8;5)	((1,1);(1,1))
(2,2,2,3,3,4,5,5,6)	(1,1,1,2;5;3)	((2,1,1);(1,1,2))
(2,2,3,3,3,3,4,4,8)	(1,1,1,1;2,2;4)	((3,1);(1,1,2))
(2,2,3,3,3,4,5,5,5)	(1,1,1;1,1,1;1)	((3,5);(2,2,4))

TABLE 6

8 AND 7 VARIABLE ORTHOGONAL DESIGNS OF ORDER 32

Orthogonal design	Construction	
	Product designs	Amicable designs
(1,1,1,1,7,7,7,7)	(1,1,7;1,1,7;7)	[(1,1);(1,1)]
(1,1,1,3,6,6,6,8)	(1,1,1;3;1)	[(1,1,2,2,2);(8)]
(1,2,3,3,4,4,5,10)	(1,2,3,3;4,5;2,5)	[(1,1);(2)]
(1,2,3,3,4,5,7,7)	(1,2,3,3;4,5;7)	[(1,1);(1,1)]
(1,2,2,2,3,11,11)	(1,2,2;2,3;11)	[(1,1);(1,1)]
(1,1,1,2,5,11,11)	(1,1,1,2;5;11)	[(1,1);(1,1)]

TABLE 7

ORTHOGONAL DESIGNS OF ORDER 64

Orthogonal design	Construction: Product designs	Construction: Amicable designs
(1,1,1,1,2,2,4,4,8,8,16,16)	(1,1,1,1,2,4;2,8;2,4,8,8)	[(1,1);(2)]
(1,1,1,2,2,3,4,6,8,12,24)	(1,1,1,2,2,4;3,8;3,6,12)	[(1,1);(2)]
(1,1,3,6,6,7,7,7,12,14)	(1,1,1,2;1,1,3;3)	[(7,1);(2,2,4)]
(1,1,2,3,4,5,5,10,11,22)	(1,1,2,3,4;11;5)	[(1,1,2);(1,1,2)]
(1,1,1,2,4,7,7,9,14,18)	(1,1,1,2,4;9;7)	[(1,1,2);(1,1,2)]
(1,2,3,3,4,5,8,8,10,20)	(1,2,3,3,4;5,8;4,5,10)	[(1,1);(2)]
(2,3,3,3,3,4,6,8,16,16)	(1,1,1,1,2;2,4;2,4,4)	[(3,1);(4)]
(3,3,3,5,5,6,8,9,10,12)	(1,1,2,3,4;3,8;5)	[(3,1);(1,1,2)]
(2,2,3,3,4,5,6,9,15,15)	(1,1,2,3;1,3,3;1)	[(3,5);(2,2,4)]
(1,1,1,2,2,9,9,9,12,18)	(1,1,1,2,2;3,4;9)	[(1,3);(1,1,2)]
(1,3,3,3,4,6,8,8,12,16)	(1,1,1,2,4;1,8;1,2,4)	[(3,1);(4)]
(4,4,5,5,5,8,11,11,11)	(1,1,1;1,1,1;1)	[(5,11);(4,4,8)]
(3,4,4,6,8,9,9,9,12)	(1,1,1;3;1)	[(9,1,2,4);(4,4,8)]
(3,3,3,4,4,8,13,13,13)	(1,1,1;1,1,1;1)	[(3,13);(4,4,8)]
(2,2,4,5,5,7,10,14,15)	(1,1,2,3;7;1)	[(5,1,2);(2,2,4)]
(2,4,6,6,7,7,9,9,14)	(1,2,3,3;9;7)	[(2,1,1);(1,1,2)]
(1,2,3,3,7,7,12,14,15)	(1,2,3,3;4,5;7)	[(1,3);(1,1,2)]

TABLE 8

14, 13 AND 12 VARIABLE ORTHOGONAL DESIGNS OF ORDER 128

Orthogonal design	Construction	
	Product designs	Amicable designs
(1,1,1,1,2,2,4,4,8,8,16,16,32,32)	(1,1,1,1,2,2,4;4,8;4,8,8,16,16)	[(1,1);(2)]
(1,1,1,2,2,3,4,6,8,12,16,24,48)	(1,1,1,2,2,3,4;6,8;6,8,12,24)	[(1,1);(2)]
(1,2,3,3,4,5,8,10,16,16,20,40)	(1,2,3,3,4,5;8,10;8,8,10,20)	[(1,1);(2)]
(1,3,3,3,4,6,8,12,16,16,24,32)	(1,1,1,2,4,8;1,16;1,2,4,8)	[(3,1);(4)]
(2,3,3,3,3,6,8,8,12,16,32,32)	(1,1,1,1,2,4;2,8;2,4,8,8)	[(3,1);(4)]

TABLE 9

11 VARIABLE ORTHOGONAL DESIGNS OF ORDER 128

Orthogonal designs	Construction: Product designs	Construction: Amicable designs
(1,1,1,1,2,4,8,16,31,31,32)	(1,1,1,2,4,8,16;1,32;31)	[(1,1);(1,1)]
(1,1,1,2,4,9,14,14,18,28,36)	(1,1,1,2,4;9;7)	[(1,1,2,4);(2,2,4)]
(1,1,2,2,3,4,5,8,10,20,72)	(1,1,2,2,3,4;5,8;5,10,36)	[(1,1);(2)]
(1,1,2,3,3,4,8,16,29,29,32)	(1,1,2,3,4,8,16;3,32;29)	[(1,1);(1,1)]
(1,1,2,3,3,6,7,7,12,14,72)	(1,1,2,3,6,12;1,24;7)	[(1,3);(1,1,2)]
(1,1,2,3,4,10,10,11,20,22,44)	(1,1,2,3,4,11;22;10)	[(1,1,2);(1,1,2)]
(1,1,2,3,4,4,7,8,14,28,56)	(1,1,2,3;7;1)	[(1,1,2,4,8);(4,4,8)]
(1,1,2,3,4,8,12,16,19,24,38)	(1,1,2,3,4,8;19;3,4,6)	[(1,1,2);(4)]
(1,1,2,3,4,8,13,13,19,26,38)	(1,1,2,3,4,8;19;13)	[(1,1,2);(1,1,2)]
(1,1,2,3,4,8,9,13,13,26,48)	(1,1,2,3,4,8;3,16;13)	[(1,3);(1,1,2)]
(1,1,2,3,6,7,7,12,14,25,50)	(1,1,2,3,6,12;25;7)	[(1,1,2);(1,1,2)]
(1,1,2,3,6,9,10,12,20,24,40)	(1,1,2,3,6,9;10,12;10,12,20)	[(1,1);(2)]
(1,1,2,3,7,7,14,18,18,21,36)	(1,1,2,3;1,1,2,3;9)	[(1,7);(2,2,4)]
(1,1,2,6,6,7,9,12,21,21,42)	(1,1,2,9;1,3,3,6;3)	[(1,7);(2,2,4)]
(1,1,2,6,7,9,12,14,24,24,28)	(1,1,2,6,7,9;12,14;12,12,14)	[(1,1);(2)]
(1,2,3,3,4,6,8,11,22,24,44)	(1,2,3,3,6,11;4,22;4,12,22)	[(1,1);(2)]
(1,3,3,4,6,8,9,16,18,24,36)	(1,3,3,6,9,18;4,36;4,8,12)	[(1,1);(2)]
(1,3,3,6,6,6,7,7,12,14,63)	(1,1,2,9;1,3,3,6;3)	[(7,1);(2,2,4)]
(1,3,3,6,7,7,9,14,18,24,36)	(1,1,2,3,6,12;1,24;7)	[(3,1);(1,1,2)]
(2,2,2,4,4,8,11,11,21,21,42)	(1,1,1,2,2,4;11;21)	[(2,1,1);(1,1,2)]
(2,2,2,4,8,15,15,16,17,17,30)	(1,1,1,2,4,8;17;15)	[(2,1,1);(1,1,2)]
(2,2,4,6,7,7,12,14,24,25,25)	(1,1,2,3,6,12;25;7)	[(2,1,1);(1,1,2)]
(2,2,4,6,8,13,13,16,19,19,26)	(1,1,2,3,4,8;19;13)	[(2,1,1);(1,1,2)]
(2,2,4,9,9,9,15,15,15,18,30)	(3,5,5,10;2,2,4,15;9)	[(3,1);(1,1,2)]
(3,3,5,5,6,9,10,15,18,18,36)	(1,1,2,3;1,1,2,3;9)	[(3,5);(2,2,4)]
(3,5,5,6,6,9,9,10,12,18,45)	(3,5,5,10;2,2,4,15;9)	[(1,3);(1,1,2)]

TABLE 10

10 VARIABLE ORTHOGONAL DESIGNS OF ORDER 128

Orthogonal design	Construction: Product designs	Construction: Amicable designs
(1,1,2,3,4,4,8,15,45,45)	(1,1,2,3;1,3,3;1)	((1,15);(4,4,8))
(1,2,2,2,5,10,20,40,23,23)	(2,2,2,5,10,20;1,40;23)	((1,1);(1,1))
(1,7,7,7,8,8,14,16,28,32)	(1,1,1,2,4;1,8;1,2,4)	((7,1);(8))
(2,2,4,5,6,9,18,23,23,36)	(2,2,4,6,9,18;5,36;23)	((1,1);(1,1))
(2,2,4,5,9,9,9,10,18,60)	(2,2,4,5,10;3,20;9)	((1,3);(1,1,2))
(2,2,4,6,7,7,9,11,14,66)	(2,2,4,6,11;3,22;7)	((1,3);(1,1,2))
(2,2,4,7,7,8,16,25,25,32)	(2,2,4,7,8,16;7,32;25)	((1,1);(1,1))
(2,2,4,7,8,9,9,18,21,48)	(2,2,4,7,8;7,16;9)	((1,3);(1,1,2))
(2,2,4,9,9,18,18,20,23,23)	(1,1,2,9,10;23;9)	((2,1,1);(1,1,2))
(2,3,8,8,9,9,16,18,22,33)	(1,3,3,6,11;2,22;8)	((3,1);(1,1,2))
(2,4,7,7,7,7,14,20,20,40)	(1,1,1,1,2;2,4;10)	((7,1);(2,2,4))
(2,6,6,9,9,12,18,20,23,23)	(1,3,3,6,10;23;9)	((2,1,1);(1,1,2))
(3,3,3,5,6,12,14,14,28,40)	(1,1,1,2,4;1,8;7)	((3,5);(2,2,4))
(3,3,4,4,6,8,9,13,39,39)	(1,1,2,3;1,3,3;1)	((3,13);(4,4,8))
(3,3,9,12,12,13,13,13,24,26)	(1,1,1,2;1,1,3;3)	((13,3);(4,4,8))
(3,7,7,8,10,10,14,20,21,28)	(1,1,2,3,4;3,8;5)	((7,1);(2,2,4))
(4,4,4,8,13,13,18,19,19,26)	(2,2,2,4,9;19;13)	((2,1,1);(1,1,2))
(4,4,5,5,7,8,10,14,15,56)	(1,1,2,3;7;1)	((5,1,2,8);(4,4,8))
(4,4,5,5,8,10,11,15,33,33)	(1,1,2,3;1,3,3;1)	((5,11);(4,4,8))
(4,4,5,8,11,11,15,15,22,33)	(1,1,2,3;1,3,3;1)	((11,5);(4,4,8))
(4,4,7,7,8,9,14,21,27,27)	(1,1,2,3;1,3,3;1)	((7,9);(4,4,8))
(4,4,7,8,9,9,18,21,21,27)	(1,1,2,3;1,3,3;1)	((9,7);(4,4,8))
(4,4,8,9,9,14,16,18,23,23)	(2,2,4,7,8;23;9)	((2,1,1);(1,1,2))
(5,5,10,10,10,11,15,20,20,22)	(1,1,2,3,4;11;5)	((5,1,2);(2,2,4))
(5,5,11,11,11,12,12,15,22,24)	(1,1,1,2;1,1,3;3)	((11,5);(4,4,8))
(5,5,5,5,6,10,12,16,32,32)	(1,1,1,1,2;2,4;2,4,4)	((5,3);(8))
(5,5,5,6,10,15,16,16,18,32)	(1,1,1,2,3;2,6;8)	((5,3);(2,2,4))
(5,5,5,7,10,10,14,18,18,36)	(1,1,1,2,2;7;9)	((5,1,2);(2,2,4))
(5,6,6,11,11,12,15,16,22,24)	(2,2,4,5,8;5,16;11)	((3,1);(1,1,2))

TABLE 11

9 AND 8 VARIABLE ORTHOGONAL DESIGNS OF ORDER 128

Orthogonal designs	Construction: Product designs	Construction: Amicable designs
(1,1,1,8,8,16,31,31,31)	(1,1,1;1,1,1;1)	((1,31);(8,8,16))
(1,2,6,6,11,11,13,26,52)	(2,6,6,13,26;1,52;11)	((1,1);(1,1))
(1,4,7,7,8,11,16,37,37)	(1,4,7,7,8;11,16;37)	((1,1);(1,1))
(1,6,7,7,9,12,24,31,31)	(1,6,7,7,12;9,24;31)	((1,1);(1,1))
(2,3,4,5,5,10,21,36,42)	(2,3,5,5,10;4,21;18,21)	((1,1);(2))
(2,4,7,7,7,7,10,14,70)	(1,1,1,1,2;2,4;10)	((7,1);(1,7))
(3,3,3,8,8,16,29,29,29)	(1,1,1;1,1,1;1)	((3,29);(8,8,16))
(4,11,11,11,11,16,16,16,32)	(1,1,1,1;4;4)	((11,1,4);(4,4,8))
(4,4,4,5,7,14,28,31,31)	(4,4,4,7,14;5,28;31)	((1,1);(1,1))
(4,4,7,8,12,14,21,29,29)	(4,4,7,8,12;14,21;29)	((1,1);(1,1))
(4,5,7,14,14,14,21,21,28)	(1,2,3,3;4,5;7)	((7,1);(2,2,4))
(5,5,10,11,15,16,20,22,24)	(1,1,2,3,4;11;2,3)	((5,1,2);(8))
(5,5,5,8,8,16,27,27,27)	(1,1,1;1,1,1;1)	((5,27);(8,8,16))
(5,5,9,10,15,16,20,24,24)	(1,1,2,3,4;3,8;2,3)	((5,3);(8))
(6,9,9,10,10,18,20,23,23)	(3,5,5,10;23;9)	((2,1,1);(1,1,2))
(1,1,1,4,28,31,31,31)	(1,1,1;1,1,1;1)	((1,31);(4,28))
(3,4,8,13,13,25,31,31)	(3,4,13,13;8,25;31)	((1,1);(1,1))
(7,7,9,11,12,24,29,29)	(7,7,9,12;11,24;29)	((1,1);(1,1))

TABLE 12

APPENDIX III

COMPUTER PROGRAMMES

1. Programmes for Chapter 5

These programmes were used on the Research School of Physical Sciences DEC-10 System and are only designed to handle full orthogonal designs; that is, orthogonal designs which contain no zeros.

Program 1 lists all full 5 variable designs of order 2^t which cannot be obtained from Lemma 5.4.1 and Lemma 5.3.2.

If we assume that all full 5 variable designs exist in order 2^{t-1} then Lemma 5.4.1 gives all full 5 variable designs of order 2^t and type (a, b, c, d, e) with more than one of a, b, c, d and e even. Therefore, Program 1 generates all full 5 variable designs of order 2^t and type (a, b, c, d, e) with exactly one of a, b, c, d and e even and outputs all of these designs which cannot be obtained from Lemma 5.3.2 into file FOR16.DAT.

In this program, "order" is 2^t, "length" is 5, and "power of 2" is t.

Program 2 determines what full 5 variable designs given in file IN.DAT cannot be obtained from a given J variable design.

The program outputs all full 5 variable designs which cannot be obtained from the given J variable design into file OUT.DAT.

In this program, "length of vector" is J and "vector" is the type of the given orthogonal design.

The program should be altered according to the comment in the program before running.

PROGRAM 1

```
      DIMENSION IB(6)
      WRITE(6,10)
10    FORMAT(' ORDER,LENGTH,POWER OF 2:-'$)
      READ(5,20)IORDER,LENGTH,IBIN
20    FORMAT(3I)
      IADD = 1
      LL = LENGTH-1
      LLLL = LL-1
      DO 30 I = 1,LLLL
30    IB(I) = 1
      IB(LL) = 1
      K = LENGTH-1
50    IB(LENGTH) = IORDER-K
55    CALL BINARY(IB,ITEST,IBIN)
      IF(ITEST.EQ.1)GOTO65
      WRITE(16,60)(IB(I),I = 1,LENGTH)
60    FORMAT(10I4)
65    IB(LL) = IB(LL) + IADD
      IB(LENGTH) = IB(LENGTH) - IADD
      IF(IB(LL).LE.IB(LENGTH))GOTO55
      DO 1000 I = LLLL,1,-1
      IB(I) = IB(I)+1
      DO 500 J = I,LL
500   IB(J) = IB(I)
      K = 0
      DO 600 J = 1,LL
600   K = K+IB(J)
      IF(IORDER-K.LT.IB(I))GOTO1000
650   NN = -1
      DO 700 J = 1,LLLL
700   NN = NN + MOD(IB(J),2)
      IF(NN)750,800,950
750   KK = LLLL-1
      GOTO900
800   KK = LLLL
900   IB(KK) = IB(KK)+1
      DO 940 J = KK,LL
940   IB(J) = IB(KK)
      GOTO650
950   NN = NN-2
      IF(NN)970,980,990
970   IADD = 2
      IB(LL) = IB(LLLL)+1-MOD(IB(LLLL),2)
      GOTO995
980   IADD = 1
      IB(LL) = IB(LLLL)
      GOTO995
990   IADD = 2
      IB(LL) = IB(LLLL)+MOD(IB(LLLL),2)
995   K = 0
      DO 996 J = 1,LL
996   K = K + IB(J)
      IF(IORDER-K.GE.IB(LL))GOTO50
1000  CONTINUE
      WRITE(16,1111)
```

Program 1 continued

```
 1111     FORMAT('999 0 0 0 0')
          STOP
          END

          SUBROUTINE BINARY(IB,ITEST,IBIN)
          DIMENSION IB(6),ID(6),IC(10)
          ITEST = 0
          IPW = IBIN-2
          IC(IPW+1) = 0
          DO 5 I = 1,5
    5     ID(I) = IB(I)
          DO 10 I = 1,IPW
   10     IC(I) = 2
          DO 20 I = 1,5
          IF(MOD(ID(I),2).EQ.1)ID(I) =  ID(I)-1
   20     CONTINUE
C
C
          DO 60 J = 1,IPW
          IDIV = 2**J
          DO 40 I = 1,5
          IF(MOD(ID(I),IDIV*2).EQ.0)GOTO40
          ID(I) = ID(I)-IDIV
          IC(J) = IC(J)-1
          IF(IC(J).LT.0)RETURN
   40     CONTINUE
          IF(MOD(IC(J),2).NE.0)GOTO100
   60     IC(J+1) = IC(J+1)+IC(J)/2
          ITEST = 1
          RETURN
  100     WRITE(6,110)
  110     FORMAT('ERROR')
          RETURN
          END
```

PROGRAM 2

```
BEGIN INTEGER I,J,K,COUNT,STP,ORDER;
I := 5;
COMMENT
**************************************
THIS PROGRAM CAN BE RUN FOR ANY ORDER
IF THE APPROPRIATE CHANGE IS MADE
IN THE NEXT LINE
************************************** ;
ORDER := 32;
STP := 0;
WRITE("LENGTH OF VECTOR");BREAK.OUTPUT;
READ(J);
    BEGIN INTEGER SUM;
    INTEGER ARRAY IA[1:I],IB[1:J];
    WRITE ("INPUT VECTOR");BREAK.OUTPUT;
    SUM := 0;
    FOR K := 1 STEP 1 UNTIL J DO
        BEGIN READ(IB[K]);SUM := SUM+IB[K] END;
    IF SUM # ORDER THEN WRITE("ERROR") ELSE
        BEGIN
        INPUT(2,"DSK");OUTPUT(3,"DSK");OPENFILE(2,"IN.DAT");
        OPENFILE(3,"OUT.DAT");OUTPUT(0,"TTY");
        SELECTINPUT(2);SELECTOUTPUT(3);COUNT := 1;
        S1:FOR K := 1 STEP 1 UNTIL I DO READ(IA[K]);
        STP := STP+1;
        IF STP REM 500 = 0 THEN
            BEGIN
            SELECTOUTPUT(0);PRINT(COUNT);
            PRINT(STP);WRITE("[C]");
            SELECTOUTPUT(3)
            END:
        IF IA[1] # 999 THEN
            BEGIN
            BOOLEAN PROCEDURE FIND(J);INTEGER J;
                BEGIN INTEGER K;BOOLEAN BLL;
                BOOLEAN PROCEDURE EQUAL(J);INTEGER J;
                    BEGIN INTEGER K;
                    IF J = 0 THEN EQUAL := TRUE ELSE
                        BEGIN
                        K := 1;
                        WHILE IF K<= I THEN
                        (IA[K] >= IB[1] OR IA[K] = 0)
                        ELSE FALSE DO K := K+1;
                        IF K<= I THEN EQUAL := FALSE ELSE
                            BEGIN
                            K := I;
                            WHILE IF K>0 THEN IA[K]#IB[J]
                            ELSE FALSE DO K := K-1;
                            IF K>0 THEN
                                BEGIN
                                IA[K] := 0;
                                EQUAL := FIND(J-1);
                                IA[K] := IB[J];
                                END
```

Program 2 continued

```
                                 ELSE EQUAL := FALSE;
                                 END;
                             END
                         END;
                     IF EQUAL(J) THEN BLL := TRUE ELSE
                         BEGIN
                         BLL := FALSE;K := I;
                         WHILE K > 0 AND NOT BLL DO
                             BEGIN
                             IF IA[K]>IB[J] THEN
                             BEGIN
                             IA[K] := IA[K]-IB[J];
                             BLL := FIND(J-1);
                             IA[K] := IA[K] + IB[J];
                             END;
                         K := K-1;
                         END;
                     END;
                 FIND := BLL;
                 END;
             IF NOT FIND(J) THEN
                 BEGIN
                 FOR K := 1 STEP 1 UNTIL I DO PRINT(IA[K],3);
                 WRITE("[C]");COUNT := COUNT +1
                 END;
             GO TO S1;
             END
         ELSE FOR K := 1 STEP 1 UNTIL I DO PRINT(IA[K]);
         WRITE("[C]");
         RELEASE(2);RELEASE(3)
         END
     END;
 WRITE("NUMBER LEFT =");PRINT(COUNT)
 END
```

2. Program for Chapter 6

Before we list the program, we need some theory in order to handle the problem efficiently and we give this theory in the form of an example.

We consider the problem of the existence of a Turyn sequence of length 13 .

If the sequences exist, we may assume they have the following form:

$$A = \{1,\ 1,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_6,\ a_5,\ a_4,\ a_3,\ 1,\ 1\}\ ,$$

$$B = \{1,\ 1,\ x_3a_3,\ x_4a_4,\ x_5a_5,\ x_6a_6,\ x_7a_7,\ x_6a_6,\ x_5a_5,\ x_4a_4,\ x_3a_3,\ 1,\ -\}\ ,$$

$$C = \{1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6,\ \bar{b}_6,\ \bar{b}_5,\ \bar{b}_4,\ \bar{b}_3,\ \bar{b}_2,\ -\}$$

$$D = \{1,\ y_2b_2,\ y_3b_3,\ y_4b_4,\ y_5b_5,\ y_6b_6,\ \bar{y}_6b_6,\ \bar{y}_5b_5,\ \bar{y}_4b_4,\ \bar{y}_3b_3,\ \bar{y}_2b_2,\ -\}\ .$$

We consider the case of $x_7 = 1$ and let $X = \{A,\ B,\ C,\ D\}$.

Now

$$\begin{aligned} N_X(10) &= 2a_3 + 2 - 2b_2 - 2y_2b_2 \\ &= 2\left[a_3+1-b_2\left(1+y_2\right)\right] \end{aligned}$$

and therefore $N_X(10) = 0$ only if

$$a_3 + 1 - b_2\left(1+y_2\right) = 0\ .$$

By considering this equation mod 4 , we obtain

$$a_3 = y_2\ .$$

By proceeding in a similar way with $N_X(i)$, $i = 9,\ 8,\ 7$ and 6 , we obtain

$$a_4 = x_3y_3\ ,$$

$$a_5 = x_4y_2y_3y_4\ ,$$

$$a_6 = x_3x_4x_5y_2y_4y_5\ ,$$

$$a_7 = x_3x_5x_6y_2y_3y_4y_5y_6\ .$$

Also, by considering $N_X(i)$, $i = 5,\ 4,\ 3,\ 2$ and 1 , we obtain

$$a_6 = x_3x_4x_5x_6y_2y_3y_5 ,$$

$$a_5 = x_3x_4y_2y_3y_4 ,$$

$$a_4 = x_3x_4y_2y_3y_5 ,$$

$$a_3 = x_3x_6y_2 ,$$

$$y_6 = 1 .$$

By combining these equations, we obtain

$$x_3 = x_6 ,$$

$$x_4 = y_2y_5 ,$$

$$x_3 = 1 ,$$

$$y_3 = y_4 ,$$

and these equations now give

$$a_3 = y_2 , \quad b_3 = y_2 , \quad y_4 = y_3 ,$$

$$a_4 = y_3 , \quad b_4 = y_2y_3y_5 , \quad y_6 = 1 , \qquad (*)$$

$$a_5 = y_5 , \quad b_5 = x_5y_5 ,$$

$$a_6 = x_5y_3 , \quad b_6 = x_5y_3 ,$$

$$a_7 = x_5y_2y_5 .$$

We now give the program for this example.

In the program we put

$$IX(2) = y_2 ,$$

$$IX(3) = y_3 ,$$

$$IX(4) = x_5 ,$$

$$IX(5) = y_5 ,$$

$$IX(6) = 1 ,$$

and enter the equations (*) into the program in the appropriate positions.

PROGRAM 3

```
      DIMENSION IA(40),IB(40),ICD(20),IT(40),IT1(40),IX(20)
C
C     **************************************
C     N IS THE SIZE OF THE LONGEST SEQUENCE
C     **************************************
      N = 13
      M = (N-1)/2-1
      MM = M+1
      MMM = N-1
      NN = M-1
      DO 10 I = 1,M
 10   IX(I) = 1
      IA(1) = 1
      IA(2) = 1
      IB(1) = 1
      IB(2) = 1
C
C
C     ***** TABLE OF A'S *****
 20   IA(3) = IX(2)
      IA(4) = IX(3)
      IA(5) = IX(5)
      IA(6) = IX(4)*IX(3)
C
C
C     CENTRE VALUE FOR A
      IA(MM+1) = IX(4)*IX(5)*IX(2)
C
C
C     ***** TABLES OF B'S *****
      IB(3) = IX(2)
      IB(4) = IX(2)*IX(5)*IX(3)
      IB(5) = IX(4)*IX(5)
      IB(6) = IX(4)*IX(3)
C
      DO 40 I = 1,MM
      IA(N+1-I) = IA(I)
 40   IB(N+1-I) = IB(I)
      IB(N) = -1
C
C     THE SIGN IN THE FOLLOWING DEPENDS ON THE A'S AND B'S
C
      IB(MM+1) = IA(MM+1)
      ISA = 0
      ISB = 0
      DO 50 I = 1,N
      ISA = ISA +IA(I)
 50   ISB = ISB + IB(I)
      ISA = ISA**2
      ISB = ISB**2
C
C     THE SUM OF THE TWO SQUARES IN THE NEXT TEST IS 4*N-2
      IF(ISA.EQ.1.AND.ISB.EQ.49)GOTO60
      IF(ISA.EQ.49.AND.ISB.EQ.1)GOTO60
      GOTO100
```

Program 3 continued

```
C
C       THE FOLLOWING ALSO DEPENDS ON THE A'S AND B'S
 60     ICD(MM) = 1
C
        DO 70 I = 1,M
 70     ICD(I) = IX(I)
C
C       ANY UNUSUAL C VALUES
        ICD(4) = IX(3)
C
        CALL INPROD(IA,IT,N)
        CALL INPROD(IB,IT1,N)
        DO 80 I = 1,MMM
 80     IT(I) = IT(I)+IT1(I)
        CALL FIND(IA,IB,IT,ICD,N)
        IF(IA(1).EQ.0)STOP
100     DO 110 I = 1,NN
        IF(IX(MM-I).EQ.1)GOTO120
110     CONTINUE
        STOP
120     IX(MM-I) = -1
        J = MM-I+1
        IF(J.EQ.MM)GOTO20
        DO 130 I = J,M
130     IX(I) = 1
        GOTO20
        END
C
C
        SUBROUTINE INPROD(IA,IT,N)
        DIMENSION IA(40),IT(40)
        IS = 0
        J = N-1
  5     DO 10 I = 1,J
 10     IS = IS +IA(I)*IA(N-J+I)
        IT(N-J) = IS
        IS = 0
        J = J-1
        IF(J.EQ.0)RETURN
        GO TO 5
        END
C
C
        SUBROUTINE FIND(IA,IB,IT,ICD,N)
        DIMENSION IA(40),IB(40),IC(40),ID(40),IT(40),ICD(40),IT1(40)
        1,IT2(4)
        M = N-1
        MM = N/2
        NN = N-2
        K = 0
        DO 20 I = 1,MM
        IF(ICD(I).EQ.1)ICD(I) = 0
        IC(I) = 1
```

Program 3 continued

```
20      ID(I) = 1
        DO 30 I = 1,MM
        IF(ICD(I).EQ.-1)GOTO40
30      CONTINUE
        GOTO45
40      ID(I) = -IC(I)
        ICD(I) = 0
45      DO 50 I = 1,MM
        IF(ICD(I).NE.0)ID(I) = -IC(I)
        IC(N-I) = -IC(I)
50      ID(N-I) = -ID(I)
        CALL INPROD(IC,IT1,M)
        CALL INPROD(ID,IT2,M)
        DO 55 I = 1,NN
        JJ = M-I
        IT1(JJ) = IT1(JJ) + IT2(JJ)+IT(JJ)
        IF(IT1(JJ).NE.0)GOTO60
55      CONTINUE
56      WRITE(6,57)(IA(I),I = 1,N)
        WRITE(6,57)(IB(I),I = 1,N)
        WRITE(6,57)(IC(I),I = 1,M)
        WRITE(6,57)(ID(I),I = 1,M)
57      FORMAT(40I3)
        RETURN
60      KK = IT1(JJ)/8
        KKK = KK*8
        IF(KKK.NE.IT1(JJ))GOTO200
        IF(K.EQ.I)GOTO100
        IF(I.GE.MM)GOTO100
        K = I
        IC(I) = -IC(I)
        ID(I) = -ID(I)
        GOTO45
100     DO 110 I = 1,MM
        IF(ICD(I).EQ.-1)GOTO120
110     CONTINUE
        RETURN
120     ICD(I) = 1
        IC(I) = -IC(I)
        ID(I) = -ID(I)
        K = 0
        L = I-1
        DO 130 II = 1,L
130     ICD(II) = -ICD(II)
        IC(II) = -IC(II)
        ID(II) = -ID(II)
        GOTO45
200     WRITE(6,210)
210     FORMAT('ERROR IN INPUTED EQUATIONS')
        IA(1) = 0
        GOTO56
        END
```

The program gives a complete search for a particular choice of x_7 and partition of $4n - 2$ into two squares.

In order to run the program for even N, only several minor changes need be made in the program.

A NON-EXISTENCE THEOREM FOR ORTHOGONAL DESIGNS

Peter J. Robinson

ABSTRACT. We show there is no orthogonal design of type $(1, 1, 1, 1, 1, n-5)$ in order n, $n > 40$.

1. *Introduction.*

An orthogonal design in order n and type $(u_1, u_2, \ldots, u_s)$ $(u_i > 0)$ on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_s\}$ such that

$$AA^T = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

Alternatively, the rows of A are formally orthogonal and each row has precisely u_i entries of the type $\pm x_i$.

In [1], where this was first defined and many examples and properties of such designs were investigated, it was mentioned that

$$A^T A = \sum_{i=1}^{s} (u_i x_i^2) I_n ,$$

and so this alternative description of A applies equally well to the columns of A. It was shown in [1] that $s \le \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d ,$$

where $n = 2^a.b$, b odd, $a = 4c + d$, $0 \le d < 4$.

Many beautiful non-existence results have been found by W. Wolfe [4] and D. Shapiro [3] who studied "algebraic" non-existence via rational matrices.

A result of Geramita and Verner [2] shows that algebraic existence is not enough to guarantee "combinatorial" existence. In this paper we give a strong combinatorial non-existence result. We use $\bar{y}$ for $-y$ and $-$ for -1.

2. *The Result.*

LEMMA.

Let
$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} x_1 & x_2 & x_3 & a \\ \bar{x}_2 & x_1 & a & \bar{x}_3 \\ \bar{x}_3 & \bar{a} & x_1 & x_2 \\ \bar{a} & x_3 & \bar{x} & x_1 \end{bmatrix};$$

then
$$P_i A_1 P_i^T = A_1, \quad i = 1, 2.$$

THEOREM. *There is no orthogonal design of type* $(1, 1, 1, 1, 1, n-5)$ *in order* n, $n > 40$.

Proof. Since this is a 6-variable design, the Radon number tells us it could only exist in orders divisible by 8.

Assume that an orthogonal design, A, of type $(1, 1, 1, 1, 1, n-5)$ exists.

First we note that it is possible to force the i^{th} diagonal 4×4 block to be of the form

$$(1)\quad \begin{bmatrix} x_1 & x_2 & x_3 & a_i \\ \bar{x}_2 & x_1 & a_i & \bar{x}_3 \\ \bar{x}_3 & \bar{a}_i & x_1 & x_2 \\ \bar{a}_i & x_3 & \bar{x}_2 & x_1 \end{bmatrix}, \quad a_i \in \{\pm x_4, \pm x_5, \pm x_6\}, i = 1, 2, \ldots, \frac{n}{4}.$$

Since x_1 appears in every diagonal position, A must be skew.

Now, if the first row of the $(i, j)^{th}$ block, $i \neq j$, is $(p\ q\ r\ s)$ then the $(i, j)^{th}$ block is

$$(2)\quad \begin{bmatrix} p & q & r & s \\ q & \bar{p} & s & \bar{r} \\ r & \bar{s} & \bar{p} & q \\ \bar{s} & \bar{r} & q & p \end{bmatrix}.$$

If $a_i = x_4$ in two of the diagonal blocks, then we may assume these are the first two blocks. But then $r(1, 5) \neq 0$. ($r(i, j)$ is the inner product between rows i and j of A.) Therefore, the variable x_4 can appear in the diagonal blocks at most twice.

Let

$$A = \begin{bmatrix} A_1 & B_1 & C_{13} & C_{14} & C_{15} & \cdots & C_{1,k-1} & C_{1k} \\ \bar{B}_1^T & A_2 & C_{23} & C_{24} & C_{25} & \cdots & C_{2,k-1} & C_{2k} \\ \bar{C}_{13}^T & \bar{C}_{23}^T & A_3 & B_2 & & & & \\ \bar{C}_{14}^T & \bar{C}_{24}^T & \bar{B}_2^T & A_4 & & & & \\ \cdot & & & & \cdot & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ \cdot & & & & & & & \\ \bar{C}_{1,k-1}^T & \bar{C}_{2,k-1}^T & & & & & A_{k-1} & B_{\frac{1}{2}k} \\ \bar{C}_{1k}^T & \bar{C}_{2k}^T & & & & \cdots & \bar{B}_{\frac{1}{2}k}^T & A_k \end{bmatrix}$$

where $k = n/4$, the B's and C's are of the form (2), $A_1, A_2, \ldots, A_{k-2}$, are of the form (1) with $a_i \neq \pm x_4$, and A_{k-1}, A_k are of the form (1).

Let $Q_i = \begin{bmatrix} I & & & & \\ & P_i & & 0 & \\ & & I & & \\ & & & \ddots & \\ 0 & & & & I \end{bmatrix}$ with P_i as in lemma.

Then $Q_i A Q_i^T$ has the same properties as A.

By applying Q_1 and Q_2 to A if necessary, we may assume that the first row of B_1 is

$$(x_4 \; b_{12} \; b_{13} \; b_{14}).$$

By using similar Q's we may assume the first row of B_i is $(x_4 \; b_{i2} \; b_{i3} \; b_{i4})$, except possibly $B_{\frac{1}{2}k}$. By taking appropriate inner products between rows of A, it can be shown that, if the first row in C_{1i} is $(p_i \; q_i \; r_i \; s_i)$ then the first row of

$$(3) \quad \left\{ \begin{array}{lll} C_{2,i+1} & \text{is } (\bar{p}_i \; q_i \; r_i \; \bar{s}_i) & \text{if } i \text{ is odd and less than } k-1 \\ C_{2,i-1} & \text{is } (p_i \; \bar{q}_i \; \bar{r}_i \; s_i) & \text{if } i \text{ is even and less than } k-1 \end{array} \right. .$$

If the first row of $B_{\frac{1}{2}k}$ is of the form $(x_4 \; b_{\frac{1}{2}k,1} \; b_{\frac{1}{2}k,2} \; b_{\frac{1}{2}k,3})$ then (3) is true for $C_{1,k-1}$, C_{1k}, $C_{2,k-1}$, C_{2k}.

If $B_{\frac{1}{2}k}$ does not contain x_4 and if the first row of C_{1i} is $(p_i \; q_i \; r_i \; s_i)$, $i = k-1, k$, then the first row of $C_{2,k-1}$ is $(s_{k-1} \; r_{k-1} \; \bar{q}_{k-1} \; \bar{p}_{k-1})$ and the first row of C_{2k} is $(\bar{s}_k \; \bar{r}_k \; q_k \; p_k)$. In either case it is easy to see that $r(1, 8)$ can be written as a sum of products of p_i's, q_i's, r_i's, s_i's taken two at a time.

For example, if $B_{\frac{1}{2}k}$ contains x_4,

$$r(1, 8) = \sum_{i=1}^{k/2} (-p_{2,i-1}s_{2i} + q_{2,i-1}r_{2i} - q_{2i}r_{2,i-1} + p_{2i}s_{2,i-1}) .$$

But, in order that A be an orthogonal design,

$$r(1, 8) = 0 .$$

Therefore $x_5 \notin \bigcup_{i=1}^{k} \{p_i, q_i, r_i, s_i\}$. Hence x_5 appears in the diagonal 8×8 blocks.

We now consider the case where x_4 appears in $B_{\frac{1}{2}k}$. Firstly we note that $A_{2i} = A_{2i-1}$ with a_{2i} replaced by $-a_{2i-1}$. We relabel the subscripts of the a_i's so that the i^{th} 8×8 block is

$$X_i = \begin{bmatrix} x_1 & x_2 & x_3 & a_i & x_4 & b_{i2} & b_{i3} & b_{i4} \\ \bar{x}_2 & x_1 & a_i & \bar{x}_3 & b_{i2} & \bar{x}_4 & b_{i4} & \bar{b}_{i3} \\ \bar{x}_3 & \bar{a}_i & x_1 & x_2 & b_{i3} & \bar{b}_{i4} & \bar{x}_4 & b_{i2} \\ \bar{a}_i & x_3 & \bar{x}_2 & x_1 & \bar{b}_{i4} & \bar{b}_{i3} & b_{i2} & x_4 \\ \bar{x}_4 & \bar{b}_{i2} & \bar{b}_{i3} & b_{i4} & x_1 & x_2 & x_3 & a_i \\ \bar{b}_{i2} & x_4 & b_{i4} & b_{i3} & \bar{x}_2 & x_1 & \bar{a}_i & \bar{x}_3 \\ \bar{b}_{i3} & \bar{b}_{i4} & x_4 & \bar{b}_{i2} & \bar{x}_3 & a_i & x_1 & x_2 \\ \bar{b}_{i4} & b_{i3} & \bar{b}_{i2} & \bar{x}_4 & a_i & x_3 & \bar{x}_2 & x_1 \end{bmatrix} .$$

In order that x_5 appears in each row and column exactly once

$$x_5 \in \{\pm a_i, \pm b_{i2}, \pm b_{i3}, \pm b_{i4}\} \quad \text{for all } i = 1, \ldots, \tfrac{1}{2}k .$$

Now if $n > 32$, that is, if there are more than 4 diagonal 8×8 blocks, $\pm x_5$ must appear in the same position in at least two blocks. We permute the blocks until these blocks are the first two.

If $\pm x_5 = a_1$ and a_2 then $r(1, 9)$ gives $a_1 = -a_2 = x_5$, but $r(1, 13)$ gives $a_1 = a_2 = x_5$, and so $\pm x_5$ cannot be a_1 and a_2. By considering similar r's it can be shown that $\pm x_5$ cannot be b_{1i} and b_{2i}, $i = 2, 3, 4$. Therefore no $(1, 1, 1, 1, 1, n-5)$ design can exist with $B_{\frac{1}{2}k}$ containing x_4 in any order greater than 32.

If $n > 40$, and $B_{\frac{1}{2}k}$ does not contain x_4, $\pm x_5$ must appear in the same position in at least two of the x_i's, $i < \frac{1}{2}k$. We permute the blocks until these are the first two blocks. Then we apply the same reasoning as before to show that there is no design of this form.

Therefore, no design of type $(1, 1, 1, 1, 1, n-5)$ can exist in any order $n > 40$.

REFERENCES

[1] Anthony V. Geramita, Joan Murphy Geramita, and Jennifer Seberry Wallis, *Orthogonal designs*, Linear and Multilinear Algebra, (to appear).

[2] A. V. Geramita, J. H. Verner, *Orthogonal designs with zero diagonal*, Canad. J. Math. (to appear).

[3] D. Shapiro, private communication, 1975.

[4] Warren W. Wolfe, *Rational quadratic forms and orthogonal designs*, J. Number Theory (to appear).

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Amicable orthogonal designs

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A powerful tool in the construction of orthogonal designs has been amicable orthogonal designs. Recent results in the construction of Hadamard matrices has led to the need to find amicable orthogonal designs A, B in order n and of types $(u_1, u_2, \ldots, u_s)$ and $(v_1, v_2, \ldots, v_r)$ respectively satisfying $A^t = -A$, $B^t = B$, and $AB^t = BA^t$ with

$$\sum_{i=1}^{s} u_i = n - 1 \text{ and } \sum_{i=1}^{r} v_i = n .$$

For simplicity, we say A, B are amicable orthogonal designs of type $(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_r)$.

We completely answer the question in order 8 by showing $(1, 2, 2, 2; 8)$, $(1, 2, 4; 2, 2, 4)$, $(2, 2, 3; 2, 6)$, $(7; 1, 7)$ and those designs derived from the above are the only possible. We use our results to obtain new orthogonal designs in order 32.

1. Introduction

DEFINITION. Two orthogonal designs, A, B, of the same order, are called *amicable orthogonal designs* if $AB^t = BA^t$.

In this paper we will be interested in amicable orthogonal designs A, B in order 8, and of types $(u_1, \ldots, u_s)$ and $(v_1, \ldots, v_r)$

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respectively, satisfying $A^t = -A$ and $B^t = B$ with $\sum_{i=1}^{s} u_i = n - 1$ and $\sum_{i=1}^{r} v_i = n$. We will say these are of type $(u_1, \ldots, u_s; v_1, \ldots, v_r)$.

In [4] Wolfe gives restrictions on the number of variables in the designs A and B. The following information is taken from Table 3 of [4]:

Number of variables in B	Maximum number of variables in A
1	4
2	3
3	3
4	3
> 4	0

For easy reference we summarize the main results of this paper.

The following amicable designs exist:

$$(1, 2, 2, 2; 8), \quad (2, 2, 3; 2, 6),$$
$$(1, 2, 4; 2, 2, 4), \quad (7; 1, 7),$$

and the following do not exist:

$$(1, 1, 5; 8), \quad (a, b; 1, 7), \quad a + b = 7, \quad a, b \neq 0,$$
$$(1, 3, 3; 8), \quad (7; 2, 2, 2, 2),$$
$$(7; 5), \quad (7; 1, 1, 6),$$
$$(2, 2, 3; 4, 4),$$

These results, together with Wolfe's results, completely answer the problem in order 8.

2. Some amicable orthogonal designs

The following lemmas can be used to construct all the designs of the required type in order 8.

For simplicity we replace -1 by $-$ and $-x_i$ by $\bar{x}_i$.

LEMMA 1. *There are amicable orthogonal designs of type* $(1, 2, 2, 2; 8)$ *in order* 8.

Proof. The following pair A, B are amicable orthogonal designs of type $(1, 2, 2, 2; 8)$:

$$A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_2 & x_4 & x_3 & x_4 \\ \bar{x}_1 & 0 & x_3 & \bar{x}_2 & x_4 & \bar{x}_2 & x_4 & \bar{x}_3 \\ \bar{x}_2 & \bar{x}_3 & 0 & x_1 & x_3 & \bar{x}_4 & \bar{x}_2 & x_4 \\ \bar{x}_3 & x_2 & \bar{x}_1 & 0 & \bar{x}_4 & \bar{x}_3 & x_4 & x_2 \\ \bar{x}_2 & \bar{x}_4 & \bar{x}_3 & x_4 & 0 & x_1 & x_2 & \bar{x}_3 \\ \bar{x}_4 & x_2 & x_4 & x_3 & \bar{x}_1 & 0 & \bar{x}_3 & \bar{x}_2 \\ \bar{x}_3 & \bar{x}_4 & x_2 & \bar{x}_4 & \bar{x}_2 & x_3 & 0 & x_1 \\ \bar{x}_4 & x_3 & \bar{x}_4 & \bar{x}_2 & x_3 & x_2 & \bar{x}_1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & - & 1 & - & - & 1 & - \\ 1 & - & 1 & 1 & - & 1 & - & - \\ - & 1 & - & - & - & 1 & - & - \\ 1 & 1 & - & 1 & 1 & 1 & - & 1 \\ - & - & - & 1 & 1 & 1 & 1 & - \\ - & 1 & 1 & 1 & 1 & - & - & - \\ 1 & - & - & - & 1 & - & - & - \\ - & - & - & 1 & - & - & - & 1 \end{bmatrix}.$$

LEMMA 2. *There are amicable orthogonal designs of type* $(1, 2, 4; 2, 2, 4)$ *in order* 8 .

Proof. Let

$$X = \begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}.$$

Put

$$A = \begin{bmatrix} x_1X & x_2Y & x_3Y & x_3Y \\ \bar{x}_2Y & x_1X & x_3Y & \bar{x}_3Y \\ \bar{x}_3Y & \bar{x}_3Y & x_1X & x_2Y \\ \bar{x}_3Y & x_3Y & \bar{x}_2Y & x_1X \end{bmatrix}$$

and

$$B = \begin{bmatrix} y_1Y & y_2Y & y_3Y & y_3Y \\ y_2Y & \overline{y}_1Y & y_3Y & \overline{y}_3Y \\ y_3Y & y_3Y & \overline{y}_2Y & \overline{y}_1Y \\ y_3Y & \overline{y}_3Y & \overline{y}_1Y & y_2Y \end{bmatrix} ;$$

then the pair A, B are amicable orthogonal designs of the required form.

LEMMA 3. *There are amicable orthogonal designs of type* $(2, 2, 3; 2, 6)$.

Proof. Put

$$A = \begin{bmatrix} 0 & x_2 & x_3 & x_3 & x_3 & x_2 & \overline{x}_1 & \overline{x}_1 \\ \overline{x}_2 & 0 & x_3 & \overline{x}_3 & x_2 & \overline{x}_3 & \overline{x}_1 & x_1 \\ \overline{x}_3 & \overline{x}_3 & 0 & x_2 & \overline{x}_1 & \overline{x}_1 & \overline{x}_2 & \overline{x}_3 \\ \overline{x}_3 & x_3 & \overline{x}_2 & 0 & \overline{x}_1 & x_1 & \overline{x}_3 & x_2 \\ \overline{x}_3 & \overline{x}_2 & x_1 & x_1 & 0 & x_2 & x_3 & x_3 \\ \overline{x}_2 & x_3 & x_1 & \overline{x}_1 & \overline{x}_2 & 0 & x_3 & \overline{x}_3 \\ x_1 & x_1 & x_2 & x_3 & \overline{x}_3 & \overline{x}_3 & 0 & x_2 \\ x_1 & \overline{x}_1 & x_3 & \overline{x}_2 & \overline{x}_3 & x_3 & \overline{x}_2 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} y_2 & y_1 & \overline{y}_2 & \overline{y}_2 & \overline{y}_2 & y_1 & \overline{y}_2 & \overline{y}_2 \\ y_1 & \overline{y}_2 & y_2 & \overline{y}_2 & \overline{y}_1 & \overline{y}_2 & \overline{y}_2 & y_2 \\ \overline{y}_2 & y_2 & \overline{y}_1 & y_2 & \overline{y}_2 & \overline{y}_2 & \overline{y}_2 & y_1 \\ \overline{y}_2 & \overline{y}_2 & y_2 & y_1 & \overline{y}_2 & y_2 & \overline{y}_1 & \overline{y}_2 \\ \overline{y}_2 & \overline{y}_1 & \overline{y}_2 & \overline{y}_2 & \overline{y}_2 & y_1 & y_2 & y_2 \\ y_1 & \overline{y}_2 & \overline{y}_2 & y_2 & y_1 & y_2 & \overline{y}_2 & y_2 \\ \overline{y}_2 & \overline{y}_2 & \overline{y}_2 & \overline{y}_1 & y_2 & \overline{y}_2 & \overline{y}_1 & \overline{y}_2 \\ \overline{y}_2 & y_2 & y_1 & \overline{y}_2 & y_2 & y_2 & \overline{y}_2 & y_1 \end{bmatrix} ;$$

then the pair A, B are amicable orthogonal designs of the required form.

LEMMA 4. *If there is a pair of amicable orthogonal designs in order* n *and of types* $(1, u_1, u_2, \ldots, u_s)$ *and* $(v_1, v_2, \ldots, v_t)$ *, then there are amicable orthogonal designs of type*

$$(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) .$$

Proof. Let X be the design of type $(1, u_1, u_2, \ldots, u_s)$, in the variables $(x_0, x_1, \ldots, x_s)$, and Y be the design of type $(v_1, \ldots, v_t)$. We can find matrices P and Q with $PP^t = QQ^t = I$ such that

$$PXQ = x_0 I + A$$

and

$$PYQ = B ,$$

where A is an orthogonal design of type $(u_1, u_2, \ldots, u_s)$ and B is a design of type $(v_1, v_2, \ldots, v_t)$. Since

$$XX^t = \left(x_0^2 + \sum_{i=1}^{s} u_i^2 x_i^2\right) I ,$$

then A is skew. We also have $XY^t = YX^t$, and hence B is symmetric and $AB^t = BA^t$.

Therefore, A, B are amicable orthogonal designs of type $(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t)$.

LEMMA 5 (Wallis [3]). *Let* $q \equiv 3 \pmod 4$ *be a prime power. Then there exists a pair of amicable orthogonal designs of order* $q + 1$ *and both of type* $(1, q)$.

The above two lemmas give the following result.

COROLLARY 6. *There are amicable orthogonal designs of type* $(7; 1, 7)$ *in order* 8 .

Other amicable orthogonal designs can be constructed from the above designs by equating variables.

3. Non-existence results

THEOREM. *There are no amicable orthogonal designs of type* (7; 5) *in order* 8 .

In order to prove this theorem, we need the following two lemmas.

LEMMA 7. *If* a *and* b *are* ± 1 , *then*

(1) $a + b \equiv ab + 1 \pmod 4$,

(2) $-a - b \equiv a + b \pmod 4$.

LEMMA 8. *Let*

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \quad and \quad B' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix} .$$

If B *is a symmetric* $(0, 1, -1)$ *matrix in order* 8 *such that* $BB^t = 5I$, *then we can find a monomial matrix,* R , *such that*

$$RBR^t = \begin{bmatrix} P & B \\ B & Q \end{bmatrix} \quad or \quad \begin{bmatrix} B_1 & I \\ I & -B_1 \end{bmatrix}$$

where P *and* Q *are monomials and* $B_1 = B$ *or* B' .

Proof. Let row i of B be $b_{i1}, b_{i2}, \ldots, b_{i8}$ and define

$$O_B(i, j) = \sum_{k=1}^{8} |b_{ik}||b_{jk}| .$$

Since $BB^T = 0$, we have $O_B(i, j) = 2$ or 4 for all i, j , $i \neq j$.

If $O_B(i, j) = 4$ for at most one j , then any column containing a zero from row i has at least six ± 1's . Therefore, for all i , there exists j_1 and j_2 such that

$$(*) \qquad O_B(i, j_1) = O_B(i, j_2) = 4 .$$

We also note that $O_B(j_1, j_2) = 4$ for the j_1 and j_2 given in (*).

Now we define an equivalence relation, $\sim$, on the rows of B as follows:

$$\text{row } i \sim \text{row } j \text{ if and only if } O_B(i, j) = 4 ,$$

and consider the equivalence classes of $\sim$.

Since each equivalence class contains at least three rows, it can be seen that there are at most two equivalences classes, each with 4 rows.

If all the rows were in the same equivalence class then $O_B(i, j) = 4$ for all i and j, which is clearly impossible.

We now consider a permutation matrix R_1, such that the first four rows of R, B are in the same equivalence class. Now let $R_1BR_1^t = B_1$.

Clearly, the first four rows of B_1 are in the same equivalence class and B_1 is symmetric, and hence it can be shown that

$$B_1 = \begin{bmatrix} A_1 & A_2 \\ A_2^t & A_3 \end{bmatrix} ,$$

where either A_1 and A_3 are symmetric Hadamard matrices in order 4 and A_2 is a monomial matrix, or A_1 and A_3 are symmetric monomial matrices and A_2 is an Hadamard matrix.

If A_2 is an Hadamard matrix, then there exist monomial matrices D and D' such that $DA_2D'^t = \mathcal{B}$.

Now let

$$R_2 = \begin{bmatrix} D & 0 \\ 0 & D' \end{bmatrix} ,$$

and therefore

$$R_2B_1R_2^t = \begin{bmatrix} P & \mathcal{B} \\ \mathcal{B} & Q \end{bmatrix} .$$

Hence

$$RBR^t = \begin{bmatrix} P & B \\ B & Q \end{bmatrix},$$

where $R = R_2R_1$. Now, if A_1 and A_3 are symmetric Hadamard matrices, then there exist monomial matrices C and D such that

$$CA_1C^t = B_1 \quad \text{and} \quad CA_2D^t = I .$$

Now, let

$$R_2' = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix};$$

then

$$R_2'B_1R_2^t = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_2^t & A_3 \end{bmatrix} \begin{bmatrix} C^t & 0 \\ 0 & D^t \end{bmatrix}$$

$$= \begin{bmatrix} B_1 & I \\ I & DA_3D^t \end{bmatrix}.$$

But $BB^t = 0$; so $DA_3D^t = -B_1$. Therefore, on putting $R = R_2'R_1$, we have the required result.

Proof of Theorem. Assume there exists amicable orthogonal designs, A and B , of type $(7; 5)$.

We may assume B is one of the forms given in Lemma 8.

Firstly we assume

$$B = \begin{bmatrix} P & B \\ B & Q \end{bmatrix}.$$

We can further assume

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{bmatrix},$$

where $a = \pm 1$, since, given any other P , either no Q can be found, or we can apply permutations to B which leave $\mathcal{B}$ fixed but transform P into one of the above forms.

Assume

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{bmatrix}$$

and let the first five rows of A be

$$\begin{matrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ \bar{a}_1 & 0 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ \bar{a}_2 & \bar{b}_2 & 0 & c_3 & c_4 & c_5 & c_6 & c_7 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 & 0 & d_4 & d_5 & d_6 & d_7 \\ \bar{a}_4 & \bar{b}_4 & \bar{c}_4 & \bar{d}_4 & 0 & e_5 & e_6 & e_7 \end{matrix} .$$

For (A, B) to be amicable orthogonal designs, AB must be symmetric.

Consider positions $(1, 4)$ and $(4, 1)$ in AB .

$$aa_2 + a_4 - a_5 - a_6 + a_7 = -a_3 + d_4 + d_5 + d_6 + d_7 .$$

Hence, by Lemma 7,

(1) $$aa_2a_3a_4a_5a_6a_7d_4d_5d_6d_7 = -1 .$$

But $-a_1b_3 - a_2c_3 + a_4d_4 + a_5d_5 + a_6d_6 + a_7d_7 = 0$ (by the orthogonality of A); that is

(2) $$a_1a_2a_4a_5a_6a_7b_3c_3d_4d_5d_6d_7 = -1 .$$

On multiplying (1) and (2) we get

(3) $$aa_1a_3b_3c_3 = 1 .$$

Now we consider positions $(2, 4)$ and $(4, 2)$ of AB . By reasoning as above, we obtain $aa_1a_3b_3c_3 = -1$ which contradicts (3).

By using similar reasoning to that of the above case, it can be shown that none of the possible B's can be used to produce amicable orthogonal designs of type $(7; 5)$ in order 8 .

LEMMA 9. *There are no amicable orthogonal designs of type* $(1, 1, 5; 8)$.

Proof. Let A, B be amicable orthogonal designs of type $(1, 1, 5; 8)$ and let A be the $(1, 1, 5)$ design in variables (x_1, x_2, x_3).

By applying various permutations on A (and B) we can assume the top left hand 4×4 block of A is

$$\begin{matrix} 0 & x_1 & x_2 & x_3 \\ \bar{x}_1 & 0 & x_3 & \bar{x}_2 \\ \bar{x}_2 & \bar{x}_3 & 0 & x_1 \\ \bar{x}_3 & x_2 & \bar{x}_1 & 0 \end{matrix} .$$

Let this block, with $x_3 = 0$, be Y and let

$$B = \begin{bmatrix} B_1 & B_2 \\ B_2^t & B_3 \end{bmatrix}$$

with B_1 and B_3 symmetric.

In order that A, B are amicable orthogonal designs YB_1 must be symmetric.

It is easy to show, however, that no such B_1 exists. Therefore, there are no amicable orthogonal designs of type $(1, 1, 5; 8)$.

The remaining results in this section will not be proved here. The proofs are longer and more involved but use the same type of reasoning as described in the above proofs. We summarize these results in the following lemma.

LEMMA 10. *There are no amicable orthogonal designs of types*

$$\begin{matrix} (1, 3, 3; 8) & , & (a, b; 1, 7) & , \quad a + b = 7 , \quad a, b \neq 0 . \\ & & (7; 2, 2, 2, 2) & , \\ (2, 2, 3; 4, 4) & , & (7; 1, 1, 6) & , \end{matrix}$$

4. Applications

In Section 2 we gave amicable orthogonal designs of type $(2, 2, 3; 2, 6)$ in order 8. By using these designs in Theorem 9 of Geramita and Wallis [1], we obtain an orthogonal design of type $(2, 3, 3, 3, 3, 6, 6, 6)$ in order 32 which gives $(3, 3, 3, 3, 20)$, $(3, 3, 6, 9, 11)$, and $(2, 3, 9, 9, 9)$ designs in order 32.

In [2] we constructed a $(1, 1, 1, 1, 1, 1, 1, 1, 8)$ and a $(1, 1, 1, 1, 1, 1, 5, 5)$ design in order 16 which give a $(1, 1, 1, 1, 2, 2, 2, 2, 16)$ and a $(1, 1, 1, 1, 2, 2, 10, 10)$ design in order 32. Hence we can construct designs of type $(1, 5, 5, 17)$, $(1, 5, 11, 11)$, and $(3, 9, 9, 9)$ in order 32. Hence we have

LEMMA 11. *In order* 32,

(i) all 5-*tuples*, $(a, b, c, d, 32-a-b-c-d)$, $0 \le a+b+c+d \le 32$, *are the types of orthogonal designs except possibly*

$(1, 3, 9, 9, 10)$,	$(1, 4, 5, 5, 17)$,
$(1, 3, 6, 11, 11)$,	$(1, 5, 6, 9, 11)$,
$(1, 5, 5, 5, 16)$,	$(1, 4, 5, 11, 11)$,
$(1, 5, 5, 10, 11)$,	$(3, 3, 4, 11, 11)$;

(ii) all 4-*tuples*, $(a, b, c, 32-a-b-c)$, $0 \le a+b+c \le 32$ *are the types of orthogonal designs in order* 32;

(iii) all 3-*tuples*, 2-*tuples, and* 1-*tuples are the types of orthogonal designs.*

References

[1] Anthony V. Geramita and Jennifer Seberry Wallis, "Some new constructions for orthogonal designs", *Combinatorics IV* (Proc. Fourth Austral. Conf., to appear).

[2] Peter J. Robinson, "Some results on orthogonal designs", submitted.

[3] Jennifer Seberry Wallis, "Constructions for amicable orthogonal designs", *Bull. Austral. Math. Soc.* 12 (1975), 179-182.

[4] Warren W. Wolfe, "Clifford algebras and amicable orthogonal designs", Queen's Mathematical Preprint No. 1974-22.

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